

Contents

Preface **xiii**

A Note for the Teacher **xv**

Chapter 1 *Matrix Operations* **1**

1. The Basic Operations 1

2. Row Reduction 9

3. Determinants 18

4. Permutation Matrices 24

5. Cramer's Rule 28

EXERCISES 31

Chapter 2 *Groups* **38**

1. The Definition of a Group 38

2. Subgroups 44

3. Isomorphisms 48

4. Homomorphisms 51

5. Equivalence Relations and Partitions 53

6. Cosets 57

7. Restriction of a Homomorphism to a Subgroup 59

8. Products of Groups 61

9. Modular Arithmetic 64

10. Quotient Groups 66

EXERCISES 69

| | | |
|------------------|--|------------|
| Chapter 3 | <i>Vector Spaces</i> | 78 |
| | 1. Real Vector Spaces | 78 |
| | 2. Abstract Fields | 82 |
| | 3. Bases and Dimension | 87 |
| | 4. Computation with Bases | 94 |
| | 5. Infinite-Dimensional Spaces | 100 |
| | 6. Direct Sums | 102 |
| | EXERCISES | 104 |
| Chapter 4 | <i>Linear Transformations</i> | 109 |
| | 1. The Dimension Formula | 109 |
| | 2. The Matrix of a Linear Transformation | 111 |
| | 3. Linear Operators and Eigenvectors | 115 |
| | 4. The Characteristic Polynomial | 120 |
| | 5. Orthogonal Matrices and Rotations | 123 |
| | 6. Diagonalization | 130 |
| | 7. Systems of Differential Equations | 133 |
| | 8. The Matrix Exponential | 138 |
| | EXERCISES | 145 |
| Chapter 5 | <i>Symmetry</i> | 155 |
| | 1. Symmetry of Plane Figures | 155 |
| | 2. The Group of Motions of the Plane | 157 |
| | 3. Finite Groups of Motions | 162 |
| | 4. Discrete Groups of Motions | 166 |
| | 5. Abstract Symmetry: Group Operations | 175 |
| | 6. The Operation on Cosets | 178 |
| | 7. The Counting Formula | 180 |
| | 8. Permutation Representations | 182 |
| | 9. Finite Subgroups of the Rotation Group | 184 |
| | EXERCISES | 188 |
| Chapter 6 | <i>More Group Theory</i> | 197 |
| | 1. The Operations of a Group on Itself | 197 |
| | 2. The Class Equation of the Icosahedral Group | 200 |
| | 3. Operations on Subsets | 203 |

- 4. The Sylow Theorems 205
- 5. The Groups of Order 12 209
- 6. Computation in the Symmetric Group 211
- 7. The Free Group 217
- 8. Generators and Relations 219
- 9. The Todd–Coxeter Algorithm 223
- EXERCISES 229

Chapter 7 Bilinear Forms **237**

- 1. Definition of Bilinear Form 237
- 2. Symmetric Forms: Orthogonality 243
- 3. The Geometry Associated to a Positive Form 247
- 4. Hermitian Forms 249
- 5. The Spectral Theorem 253
- 6. Conics and Quadrics 255
- 7. The Spectral Theorem for Normal Operators 259
- 8. Skew-Symmetric Forms 260
- 9. Summary of Results, in Matrix Notation 261
- EXERCISES 262

Chapter 8 Linear Groups **270**

- 1. The Classical Linear Groups 270
- 2. The Special Unitary Group SU_2 272
- 3. The Orthogonal Representation of SU_2 276
- 4. The Special Linear Group $SL_2(\mathbb{R})$ 281
- 5. One-Parameter Subgroups 283
- 6. The Lie Algebra 286
- 7. Translation in a Group 292
- 8. Simple Groups 295
- EXERCISES 300

Chapter 9 Group Representations **307**

- 1. Definition of a Group Representation 307
- 2. G-Invariant Forms and Unitary Representations 310
- 3. Compact Groups 312
- 4. G-Invariant Subspaces and Irreducible Representations 314

- 5. Characters 316
- 6. Permutation Representations and the Regular Representation 321
- 7. The Representations of the Icosahedral Group 323
- 8. One-Dimensional Representations 325
- 9. Schur's Lemma, and Proof of the Orthogonality Relations 325
- 10. Representations of the Group SU_2 330
- EXERCISES 335

Chapter 10 Rings 345

- 1. Definition of a Ring 345
- 2. Formal Construction of Integers and Polynomials 347
- 3. Homomorphisms and Ideals 353
- 4. Quotient Rings and Relations in a Ring 359
- 5. Adjunction of Elements 364
- 6. Integral Domains and Fraction Fields 368
- 7. Maximal Ideals 370
- 8. Algebraic Geometry 373
- EXERCISES 379

Chapter 11 Factorization 389

- 1. Factorization of Integers and Polynomials 389
- 2. Unique Factorization Domains, Principal Ideal Domains, and Euclidean Domains 392
- 3. Gauss's Lemma 398
- 4. Explicit Factorization of Polynomials 402
- 5. Primes in the Ring of Gauss Integers 406
- 6. Algebraic Integers 409
- 7. Factorization in Imaginary Quadratic Fields 414
- 8. Ideal Factorization 419
- 9. The Relation Between Prime Ideals of R and Prime Integers 424
- 10. Ideal Classes in Imaginary Quadratic Fields 425
- 11. Real Quadratic Fields 433

12. Some Diophantine Equations 437
EXERCISES 440

Chapter 12 Modules 450

1. The Definition of a Module 450
 2. Matrices, Free Modules, and Bases 452
 3. The Principle of Permanence of Identities 456
 4. Diagonalization of Integer Matrices 457
 5. Generators and Relations for Modules 464
 6. The Structure Theorem for Abelian Groups 471
 7. Application to Linear Operators 476
 8. Free Modules over Polynomial Rings 482
- EXERCISES 483

Chapter 13 Fields 492

1. Examples of Fields 492
 2. Algebraic and Transcendental Elements 493
 3. The Degree of a Field Extension 496
 4. Constructions with Ruler and Compass 500
 5. Symbolic Adjunction of Roots 506
 6. Finite Fields 509
 7. Function Fields 515
 8. Transcendental Extensions 525
 9. Algebraically Closed Fields 527
- EXERCISES 530

Chapter 14 Galois Theory 537

1. The Main Theorem of Galois Theory 537
 2. Cubic Equations 543
 3. Symmetric Functions 547
 4. Primitive Elements 552
 5. Proof of the Main Theorem 556
 6. Quartic Equations 560
 7. Kummer Extensions 565
 8. Cyclotomic Extensions 567
 9. Quintic Equations 570
- EXERCISES 575

| | | |
|--|----------------------------------|-----|
| <i>Appendix</i> | <i>Background Material</i> | 585 |
| | 1. Set Theory | 585 |
| | 2. Techniques of Proof | 589 |
| | 3. Topology | 593 |
| | 4. The Implicit Function Theorem | 597 |
| | EXERCISES | 599 |
| <i>Notation</i> | | 601 |
| <i>Suggestions for Further Reading</i> | | 603 |
| <i>Index</i> | | 607 |