

# Contents

Notation	xi
Introduction	xiii
<b>CHAPTER 1</b>	
<b>Character Sums</b>	<b>1</b>
1. Character Sums over Finite Fields	1
2. Stickelberger's Theorem	6
3. Relations in the Ideal Classes	14
4. Jacobi Sums as Hecke Characters	16
5. Gauss Sums over Extension Fields	20
6. Application to the Fermat Curve	22
<b>CHAPTER 2</b>	
<b>Stickelberger Ideals and Bernoulli Distributions</b>	<b>26</b>
1. The Index of the First Stickelberger Ideal	27
2. Bernoulli Numbers	32
3. Integral Stickelberger Ideals	43
4. General Comments on Indices	48
5. The Index for $k$ Even	49
6. The Index for $k$ Odd	50
7. Twistings and Stickelberger Ideals	51
8. Stickelberger Elements as Distributions	53
9. Universal Distributions	57
10. The Davenport–Hasse Distribution	61
Appendix. Distributions	65

CHAPTER 3

Complex Analytic Class Number Formulas 69

- 1. Gauss Sums on  $\mathbf{Z}/m\mathbf{Z}$  69
- 2. Primitive  $L$ -series 72
- 3. Decomposition of  $L$ -series 75
- 4. The  $(\pm 1)$ -eigenspaces 81
- 5. Cyclotomic Units 84
- 6. The Dedekind Determinant 89
- 7. Bounds for Class Numbers 91

CHAPTER 4

The  $p$ -adic  $L$ -function 94

- 1. Measures and Power Series 95
- 2. Operations on Measures and Power Series 101
- 3. The Mellin Transform and  $p$ -adic  $L$ -function 105
  - Appendix. The  $p$ -adic Logarithm 111
- 4. The  $p$ -adic Regulator 112
- 5. The Formal Leopoldt Transform 115
- 6. The  $p$ -adic Leopoldt Transform 117

CHAPTER 5

Iwasawa Theory and Ideal Class Groups 123

- 1. The Iwasawa Algebra 124
- 2. Weierstrass Preparation Theorem 129
- 3. Modules over  $\mathbf{Z}_p[[X]]$  131
- 4.  $\mathbf{Z}_p$ -extensions and Ideal Class Groups 137
- 5. The Maximal  $p$ -abelian  $p$ -ramified Extension 143
- 6. The Galois Group as Module over the Iwasawa Algebra 145

CHAPTER 6

Kummer Theory over Cyclotomic  $\mathbf{Z}_p$ -extensions 148

- 1. The Cyclotomic  $\mathbf{Z}_p$ -extension 148
- 2. The Maximal  $p$ -abelian  $p$ -ramified Extension of the Cyclotomic  $\mathbf{Z}_p$ -extension 152
- 3. Cyclotomic Units as a Universal Distribution 157
- 4. The Iwasawa–Leopoldt Theorem and the Kummer–Vandiver Conjecture 160

CHAPTER 7

Iwasawa Theory of Local Units 166

- 1. The Kummer–Takagi Exponents 166
- 2. Projective Limit of the Unit Groups 175
- 3. A Basis for  $U(\chi)$  over  $\Lambda$  179
- 4. The Coates–Wiles Homomorphism 182
- 5. The Closure of the Cyclotomic Units 186

## CHAPTER 8

<b>Lubin–Tate Theory</b>	<b>190</b>
1. Lubin–Tate Groups	190
2. Formal $p$ -adic Multiplication	196
3. Changing the Prime	200
4. The Reciprocity Law	203
5. The Kummer Pairing	204
6. The Logarithm	211
7. Application of the Logarithm to the Local Symbol	217

## CHAPTER 9

<b>Explicit Reciprocity Laws</b>	<b>220</b>
1. Statement of the Reciprocity Laws	221
2. The Logarithmic Derivative	224
3. A Local Pairing with the Logarithmic Derivative	229
4. The Main Lemma for Highly Divisible $x$ and $\alpha = x_n$	232
5. The Main Theorem for the Symbol $\langle x, x_n \rangle_n$	236
6. The Main Theorem for Divisible $x$ and $\alpha = \text{unit}$	239
7. End of the Proof of the Main Theorems	242

## CHAPTER 10

<b>Measures and Iwasawa Power Series</b>	<b>244</b>
1. Iwasawa Invariants for Measures	245
2. Application to the Bernoulli Distributions	251
3. Class Numbers as Products of Bernoulli Numbers	258
Appendix by L. Washington: Probabilities	261
4. Divisibility by $l$ Prime to $p$ : Washington's Theorem	265

## CHAPTER 11

<b>The Ferrero–Washington Theorems</b>	<b>269</b>
1. Basic Lemma and Applications	269
2. Equidistribution and Normal Families	272
3. An Approximation Lemma	276
4. Proof of the Basic Lemma	277

## CHAPTER 12

<b>Measures in the Composite Case</b>	<b>280</b>
1. Measures and Power Series in the Composite Case	280
2. The Associated Analytic Function on the Formal Multiplicative Group	286
3. Computation of $L_p(1, \chi)$ in the Composite Case	291

CHAPTER 13	
Divisibility of Ideal Class Numbers	295
1. Iwasawa Invariants in $\mathbf{Z}_p$ -extensions	295
2. CM Fields, Real Subfields, and Rank Inequalities	299
3. The $l$ -primary Part in an Extension of Degree Prime to $l$	304
4. A Relation between Certain Invariants in a Cyclic Extension	306
5. Examples of Iwasawa	310
6. A Lemma of Kummer	312
CHAPTER 14	
$p$ -adic Preliminaries	314
1. The $p$ -adic Gamma Function	314
2. The Artin–Hasse Power Series	319
3. Analytic Representation of Roots of Unity	323
Appendix: Barsky’s Existence Proof for the $p$ -adic Gamma Function	325
CHAPTER 15	
The Gamma Function and Gauss Sums	329
1. The Basic Spaces	330
2. The Frobenius Endomorphism	336
3. The Dwork Trace Formula and Gauss Sums	341
4. Eigenvalues of the Frobenius Endomorphism and the $p$ -adic Gamma Function	343
5. $p$ -adic Banach Spaces	348
CHAPTER 16	
Gauss Sums and the Artin–Schreier Curve	360
1. Power Series with Growth Conditions	360
2. The Artin–Schreier Equation	369
3. Washnitzer–Monsky Cohomology	374
4. The Frobenius Endomorphism	378
CHAPTER 17	
Gauss Sums as Distributions	381
1. The Universal Distribution	381
2. The Gauss Sums as Universal Distributions	385
3. The $L$ -function at $s = 0$	389
4. The $p$ -adic Partial Zeta Function	391

## APPENDIX BY KARL RUBIN

<b>The Main Conjecture</b>	<b>397</b>
Introduction	397
1. Setting and Notation	397
2. Properties of Kolyvagin's "Euler System"	399
3. An Application of the Chebotarev Theorem	401
4. Example: The Ideal Class Group of $\mathbf{Q}(\mu_p)^+$	403
5. The Main Conjecture	405
6. Tools from Iwasawa Theory	406
7. Proof of Theorem 5.1	411
8. Other Formulations and Consequences of the Main Conjecture	415
<b>Bibliography</b>	<b>421</b>
<b>Index</b>	<b>431</b>