Table of Contents

Chapter I: The Axioms 1

I.1 Enumeration of the Axioms I-XIII. Remarks.

I.2 Interpretation of the axioms.

Chapter II: Geometrical character of L and the ring $\mathfrak{R}$ 8

II.1 Verification that L fulfills the Axioms I°-VI° of a discrete (projective) or continuous geometry.

II.2 Introduction of the ring $\mathfrak{R}$ (Theorem I, Assumption Δ).

Chapter III: Unitary transformations 23

III.1 Elementary properties.

III.2 P(1,a) = D(a). The first transitivity Lemma. (Theorem II).

III.3 Partial unitarity. Transitivity properties. (Theorem III).

Appendix III Elimination of Axiom $XIII_1$ for $L = L_{\infty}$.

Appendix III′ Simultaneous observability (Theorems I_b, II_b).

Chapter IV: Definiteness 41

IV.1 Orthogonal systems and the Hermitian matrix scheme. (Theorems IV, V).

IV.2 Definiteness (Theorem VI).

IV.3 The characteristic ρ_0 of $\mathfrak{Z}$ is 0 (Theorem VII).

IV.4 The relations $x < y$ and $\|x\| \leq k$, and the notion of boundedness (Theorems VIII, IX, X).

IV.5 Functional calculus (First part).

Chapter V: Real traces 67

V.1 Definition of the real trace: t(xy) = t(yx), t(g) = R(g) for h. ip. g..

V.2 Extension properties of the real trace (Theorem XI).

V.3 The trace metric (Theorem XII).

V.4 Approximations.

Chapter VI: Real traces and transition probabilities 84

VI.1 $P(a,b) = \frac{D(b)}{D(a)}$ for $b \leq a$. The quantities $\pi_h(a,b)$, $h = 1, 2$ (Theorem XIII).

VI.2 The function $\tau(x)$ with $\tau(fgf) = \pi_h((f)_r, (g)_r)$.

VI.3 The real trace $\bar{t}(x)$ in $\mathfrak{R}$ with $\bar{t}(fg) = \pi_h((f)_r,(g)_r)$ (First part).

VI.4 The real trace $\ddot{t}(x)$ in $\mathfrak{R}$ with $\bar{t}(fg) = \pi_h((f)_r,(g)_r)$ (Second part) (Theorem XIV).

Chapter VII: Completeness properties. Uniqueness of the real trace. The center 106

VII.1 Completeness properties (Theorem XV).

VII.2 Functional calculus (Second part) (Theorem XVI).

VII.3 Spectral decomposition (Theorems XVII, XVIII).

VII.4 Uniqueness of the real trace (Theorems XIX, XX).

VII.5 Determination of the center (Theorem XXI).

Chapter VIII: The operatorial description in a unitary space 136

VIII.1 Construction of the unitary space $\mathfrak{H}$ (Theorems XXII, XXIII).

VIII.2 The operators R_u, L_u, C in $\mathfrak{H}$.

VIII.3 Properties of the R_u, L_u, $\bar{R}_u$, $\bar{L}_u$.

VIII.4 The rings $\mathfrak{m}$, $\mathfrak{n}$ (Theorem XXIV).

VIII.5 The systems $u(\mathfrak{m})$, $u(\mathfrak{n})$ (Theorems XXV, XXVI, XXVII).

VIII.6 $\mathfrak{m}$, $\mathfrak{n}$ as factors (Theorem XXVIII).

Appendix VIII The discrete cases L_N, N = 4, 5, ... (Theorems I_c, II_c).

Chapter IX: Construction of all systems L 189

IX.1 Rings of operators in a unitary space.

Appendix IX, added February 1980 by Israel Halperin

Appendix IX_1 The L which satisfy Axioms I-XIII and Assumption Δ, are precisely all projection geometries of factors in Case (I_N), N = 4, 5, ..., or in Case (II_1), up to $(\prec, \perp, P)$ isomorphisms (Theorem I_d).

Appendix IX_2 Von Neumann's stronger Axiom IX_o gives just those L which are $(\prec, \perp, P)$ isomorphic to projection geometries of factors in Case (I_N), N = 4, 5, ... (Theorem II_d).

Appendix IX_3 Axiom X is superfluous.

Appendix IX_4 Operator algebra isomorphism and $(\prec, \perp, P)$ isomorphism (Theorem III_d).

Appendix IX_5 Normal form for the discrete $L(\mathfrak{m})$, $\mathfrak{m}$ in Case (I_N), N = 4, 5, ...