

Contents

I. Analytic tools

Chapter 1. Bernoulli polynomials and Bernoulli numbers	1
1. The binomial coefficients	1
2. The Bernoulli polynomials	4
3. Zeros of the Bernoulli polynomials	7
4. The Bernoulli numbers	9
5. The von Staudt-Clausen theorem	10
6. A multiplication formula for the Bernoulli polynomials	13
Chapter 2. The Euler-MacLaurin sum formula	14
7. Use of the Bernoulli polynomials	14
8. Fourier expansions of the Bernoulli polynomials	15
9. Sums of reciprocal powers	16
10. The generating function of the Bernoulli numbers	18
11. Tangent and cotangent coefficients	19
12. A theorem by Frobenius about the numerators of the Bernoulli numbers	20
13. The generating function of the Bernoulli polynomials	23
14. The secant coefficients or Euler numbers	24
15. Stirling's formula	26
16. A further application	28
17. A historical remark	29
Chapter 3. The Γ -function and Mellin's theorem	30
18. Definition of the Γ -function	30
19. Functional equations of $\Gamma(s)$	32
20. Application of the Euler-MacLaurin sum formula	34
21. Asymptotic behavior of $\Gamma(s)$	37
22. A lemma	39
23. The Mellin formula	41
24. Hankel's formula	45
25. An application to Bessel functions	46
26. The Fourier integral	47
27. Mellin's formulae	52
28. Some further examples of Mellin's formulae	53
Chapter 4. The Phragmén-Lindelöf theorem	58
29. The main theorem	58
30. A theorem of the Phragmén-Lindelöf type for subharmonic functions	60
31. The Poisson integral formula for a strip	61
32. A lemma	63
33. A generalization of the Phragmén-Lindelöf theorem	66
34. Applications to the Γ -function	68

Chapter 5. The Poisson sum formula and applications	70
35. The theorem	70
36. Application: A transformation formula for a ϑ -function	75
37. Lipschitz's formula	77
II. Special functions	
Chapter 6. The Riemann ζ -function	80
38. Definition of the ζ -function and its analytic continuation	80
39. Two special integrals	82
40. Riemann's functional equation for $\zeta(s)$	83
41. Another proof for the functional equation of $\zeta(s)$	86
42. Connection between the ζ -function and a ϑ -function	88
43. Estimation of $\zeta(s)$ in a vertical strip	91
Chapter 7. About the prime-number theorem and the zeros of the ζ -function	93
44. The Euler product	93
45. The borders of the critical strip are free of zeros of $\zeta(s)$	94
46. Preparation for the proof of the prime-number theorem	95
47. A lemma	97
48. Expression of a function $\Psi(x)$ connected with $\psi(x)$ by means of an integral	98
49. Some estimates for $\zeta(s)$, $\zeta'(s)$, $1/\zeta(s)$	99
50. The prime-number theorem	101
51. The error term in the prime-number theorem	104
52. Carathéodory's lemma	105
53. Application of Carathéodory's lemma	107
54. The error term $r(x)$	108
55. Existence of infinitely many non-trivial zeros	111
56. Additional remarks	113
57. Dirichlet series and the best order of the error term in the prime-number theorem	113
Chapter 8. The Eisenstein series	117
58. Definition of the Eisenstein series and of $\wp(u)$	117
59. Expansion of $\wp(u)$ in a Laurent series	119
60. Lambert series	121
61. Some arithmetical consequences	123
62. Modular forms	124
63. Definition of $G_2(\omega_1, \omega_2)$	126
64. The modular invariance of $G_2(\omega_1, \omega_2)$	132
65. Dedekind function $\eta(\tau)$ and the discriminant $\Delta(\tau)$	133
Chapter 9. The transformation of $\log \eta(\tau)$ and the theory of the Dedekind sums	137
66. A formula of Iseki	137
67. Application of Iseki's formula to the transformation of $\log \eta(\tau)$	143
68. The Dedekind sums	145
69. The formula of reciprocity of the Dedekind sums	146
70. A direct proof of the reciprocity formula for Dedekind sums	148
71. Composition of modular transformations of $\eta(\tau)$	150
72. A group-theoretical remark	154
73. The Dedekind sums and the Jacobi residue symbol	157
74. Again the transformation of $\eta(\tau)$	160

Chapter 10. The ϑ -functions	164
75. Introduction of the ϑ -functions	164
76. Definition of the ϑ -functions	166
77. Zeros of the ϑ -functions	167
78. Product expansions of the ϑ -functions	169
79. Transformation of the ϑ -functions	173
80. Transformation of $\vartheta_1(v \tau)$, continued	177
81. Transformation of $\vartheta_2(v \tau)$, $\vartheta_3(v \tau)$, $\vartheta_4(v \tau)$	181
Chapter 11. Elliptic functions and their applications to number theory	183
82. Construction of elliptic functions from the ϑ -functions	183
83. Sums of four squares	184
84. Sums of two squares	186
85. Lambert series for $f_\alpha(v)$	189
86. Lambert series for $f_\alpha^2(v)$	191
87. Some addition formulae for ϑ -functions	193
88. Formulae of differentiation	195
89. Even powers of ϑ_3 expressed by derivatives of $f_\alpha(v)$ and $f_\alpha^2(v)$	196
90. Lambert series for the even powers of ϑ_3	197
91. Sums of an even number of squares	198
92. Discussion of the foregoing results	200
93. Further discussion of $\rho(n)$	204

III. Formal power series

Chapter 12. Formal power series and the theory of partitions ...	209
94. Introduction and definitions	209
95. Some elementary identities	213
96. Partitions with restricted size or number of parts	216
97. Some similar theorems	219
98. Unrestricted partitions	223
99. Formal differentiation and its application	228
100. Jacobi's triple product	231
101. Another proof of the pentagonal numbers theorem	234
102. A Jacobi formula	235
103. An identity of Euler	236
Chapter 13. Ramanujan's congruences and identities	237
104. Some divisibility properties of $p(n)$	237
105. Two Ramanujan identities	241
106. Relations between the G_s , H_s and φ	244
107. The Rogers-Ramanujan identities. Introductory remarks	246
108. Arithmetical statement of the identities	247
109. Reformulation of the problem	249
110. The Gaussian polynomials	250
111. Schur's functions	252
112. Linear combinations of Schur's functions	254
113. Determination of $D_1(x)$ and $D_2(x)$	258
114. A digression, concerning a further proof of the pentagonal number theorem	259
115. A further remark	263

IV. The circle method

Chapter 14. Analytic theory of partitions	264
116. A Cauchy integral and a special path of integration	264
117. An expression for $p(n)$	267
118. Application of the transformation formula for $\eta(\tau)$	268
119. Estimates and evaluations	270
120. Continuation of estimates and evaluations. The final formula for $p(n)$	271
121. A partial sum with error term	275
122. Discussion of the sums $A_k(n)$, A new expression for ω_{hk}	279
123. A lemma by Whiteman and the Selberg sum	280
124. Different cases of $B_k(\nu)$ according to k	283
125. Multiplicativity of $B_k(\nu)$	285
126. Evaluation of $B_k(\nu)$ for a prime power	287
127. Estimations of $A_k(n)$	291
128. The generating function $f(x)$ for $p(n)$	292
129. Discussion of $\Phi_k(z)$	294
130. Decomposition of $f(x)$ into partial fractions	299
Chapter 15. Application of the circle method to modular forms of positive dimension	303
131. Generalized modular forms	303
132. Computation of the coefficients of the modular form	306
133. Estimations	308
134. The final formula for the coefficients	310
135. The series for the modular form $F(\tau)$	311
Editor's notes	314
Bibliography	315
Index	319