

Contents

| | | |
|---------|---|----|
| Preface | | xv |
| 1 | Estimates | 1 |
| 1. | Introduction | 1 |
| 2. | Comparison of the Sum $\sum_{k=0}^N f(k)$ with the Integral $\int_0^N f(x)dx$ | 1 |
| 3. | Lattice Points | 2 |
| 4. | Convergence | 3 |
| 5. | Euler's Constant | 4 |
| 6. | Inequalities | 5 |
| 7. | Continuous Version | 6 |
| 8. | A Technique of Shnirelman | 7 |
| 9. | Summation by Parts | 7 |
| 10. | The Gaussian Integral | 8 |
| 11. | The Hecke Integral | 9 |
| | Miscellaneous Exercises | 11 |
| | Bibliography and Comments | 12 |
| 2 | Transforms | 15 |
| 1. | Introduction | 15 |
| 2. | Fourier Series | 15 |
| 3. | Fejer's Theorem | 18 |
| 4. | Parseval's Theorem | 19 |
| 5. | Fourier Series for the Fractional Part | 20 |
| 6. | Finite Fourier Series | 21 |
| 7. | The Fourier Transform | 22 |
| 8. | Parseval-Plancherel Relation | 22 |
| 9. | The Laplace Transform | 22 |
| 10. | The Product Formula | 24 |
| 11. | The Identity of Lipshitz | 24 |

| | |
|---|-----------|
| 12. The Mellin Transform | 25 |
| 13. The Product Formula | 25 |
| Miscellaneous Exercises | 26 |
| Bibliography and Comments | 26 |
| 3 Congruences | 31 |
| 1. Introduction | 31 |
| 2. The Congruence Notation | 31 |
| 3. Fermat's Theorem | 32 |
| 4. The Linear Equation | 34 |
| 5. The Quadratic Equation | 35 |
| 6. The General Polynomial Equation | 35 |
| 7. Connection with Trigonometric Sums | 36 |
| Miscellaneous Exercises | 38 |
| Bibliography and Comments | 41 |
| 4 The Γ Function | 45 |
| 1. Introduction | 45 |
| 2. The Basic Functional Equation | 45 |
| 3. $\Gamma(s)$ as a Real Function | 47 |
| 4. Laplace's Asymptotic Evaluation of an Integral | 47 |
| 5. The Representation of Weierstrass | 49 |
| 6. Half the Sine Function | 50 |
| 7. The Duplication Formula | 51 |
| 8. The Method of Stationary Phase | 52 |
| 9. The Beta Function | 52 |
| 10. The Integral of Siegel | 53 |
| Miscellaneous Exercises | 54 |
| Bibliography and Comments | 55 |
| 5 Riemann Zeta Function | 57 |
| 1. Introduction | 57 |
| 2. Dirichlet Series | 57 |
| 3. The Euler Product | 58 |
| 4. The Möbius Function | 59 |
| 5. The Möbius Inversion Formula | 59 |
| 6. The Squarefree Function | 60 |
| 7. Multiplicative Functions | 60 |
| 8. Analytic Continuation | 61 |
| 9. The Functional Equation | 63 |
| 10. The Riemann Hypothesis | 64 |

| | |
|--|-----------|
| 11. Asymptotic Behavior | 64 |
| 12. Uses of the ζ Function | 66 |
| 13. Mean Values of the ζ Function | 67 |
| 14. The Prime Number Theorem | 67 |
| Miscellaneous Exercises | 69 |
| Bibliography and Comments | 72 |
| 6 The Poisson Summation Formula | 77 |
| 1. Introduction | 77 |
| 2. Fourier Series | 77 |
| 3. The Poisson Summation Formula | 79 |
| 4. Some Simple Sufficient Conditions | 81 |
| 5. Application to the Theta Function | 81 |
| 6. Multi-dimensional Version | 82 |
| 7. Interesting Domains | 82 |
| 8. The Evaluation of $\int_{-\infty}^{\infty} e^{-x, Ax} dx$ | 82 |
| 9. The General Modular Transformation | 83 |
| 10. Sums of Squares | 83 |
| 11. Partition Functions | 84 |
| 12. Gaussian Sums | 84 |
| 13. Einstein Series | 86 |
| 14. Functional Equations | 87 |
| 15. Use of the Laplace Transform | 87 |
| Bibliography and Comments | 89 |
| 7 Functional Equations | 95 |
| 1. Introduction | 95 |
| 2. Functional Equations Satisfied by Theta Functions | 95 |
| 3. Determination of $\theta(z, t)$ | 97 |
| 4. The Modular Transformation | 97 |
| 5. The Determination of the Multiplier | 97 |
| 6. Partial Differential Equations | 98 |
| 7. Generalizations | 98 |
| 8. k -Dimensional Linear Spaces | 98 |
| 9. The Case $k = 2$ | 99 |
| 10. Complex Multiplication | 100 |
| 11. Doubly-Periodic Functions | 100 |
| Miscellaneous Exercises | 101 |
| Bibliography and Comments | 103 |

| | | |
|-----|---|-----|
| 8 | The Euler φ Function | 109 |
| 1. | Introduction | 109 |
| 2. | The Multiplicative Property | 109 |
| 3. | The Euler Product | 110 |
| 4. | The Mean Value of $\varphi(n)$ | 111 |
| 5. | The Mean Value of $\varphi(n)^2$ | 112 |
| 6. | An Alternate Method | 113 |
| 7. | The Mean Value of $\varphi(p(n))$ | 114 |
| 8. | The Ramanujan Function | 114 |
| | Miscellaneous Exercises | 116 |
| | Bibliography and Comments | 117 |
| 9 | The Divisor Function | 125 |
| 1. | Introduction | 125 |
| 2. | The Divisor Function | 125 |
| 3. | The Mean Value of $d(n)$ | 126 |
| 4. | A Simple Geometric Approach | 127 |
| 5. | Analytic Equivalent | 128 |
| 6. | The Dirichlet Divisor Problem | 128 |
| 7. | The Mean Value of $d(n)^2$ | 129 |
| 8. | Connection with Diophantine Equations | 129 |
| 9. | The Perron Sum Formula | 130 |
| 10. | Logarithmic Summability | 132 |
| 11. | Asymptotic Behavior | 133 |
| 12. | The Mean Value of $d(an^2 + bn + c)$ | 135 |
| 13. | The Mean Value of $d(p(n))$ | 135 |
| 14. | Ramanujan Expansions | 136 |
| | Miscellaneous Exercises | 137 |
| | Bibliography and Comments | 138 |
| 10 | The Squarefree Problem | 143 |
| 1. | Introduction | 143 |
| 2. | The Generating Function | 143 |
| 3. | The Mean Value of $\mu^2(n)$ | 144 |
| 4. | The Mean Value of $\mu^2(n^2 + 1)$ | 145 |
| 5. | The Pell Equation | 145 |
| 6. | The Number of Squarefree Numbers of the Form $n^2 + 1$ | 146 |
| 7. | The Number of Squarefree Numbers in a General Polynomial Sequence | 149 |
| | Miscellaneous Exercises | 150 |
| | Bibliography and Comments | 151 |

| | | |
|----|--|-----|
| 11 | The Prime Divisor Function, Selberg's Sieve Method, and Algebraic Independence | 155 |
| 1. | Introduction | 155 |
| 2. | The Prime Divisor Function | 155 |
| 3. | The Mean Value $2\omega(n)$ | 156 |
| 4. | The Mean Value $\omega(n)$ | 156 |
| 5. | The Order of Magnitude of $\sum_{n=1}^N \omega^2(n)$ | 157 |
| 6. | The Average order of $\omega(g(n))$ | 158 |
| 7. | The Sieve Method of Selberg | 159 |
| 8. | The Algebraic Independence of Arithmetic Functions | 163 |
| | Bibliography and Comments | 171 |
| 12 | Tauberian Theorems | 173 |
| 1. | Introduction | 173 |
| 2. | The Tauberian Theorem of Hardy and Littlewood | 173 |
| 3. | Continuous Version | 174 |
| 4. | The Mean Value of $\zeta(1/2 + it)^2$ | 175 |
| 5. | The Error Term in $\sum_{n=1}^N d(n)$ | 176 |
| 6. | The Mean Value of $ \zeta(1/2 + it) ^4$ | 183 |
| 7. | The Erdős-Selberg Method for the Prime Number Theorem | 183 |
| | Miscellaneous Exercises | 184 |
| | Bibliography and Comments | 185 |
| | Name Index | 189 |
| | Subject Index | 195 |