

CONTENTS

FOREWORD TO THE FIRST EDITION	xiii
FOREWORD TO THE SECOND EDITION	xvii
INTRODUCTION	xix

I. INTEGRALS OF THE CAUCHY TYPE

§ 1. DEFINITION OF THE CAUCHY TYPE INTEGRAL AND EXAMPLES	1
§ 2. FUNCTIONS SATISFYING THE HÖLDER CONDITION	5
2.1. Definition and properties (5). 2.2. Functions of many variables (7).	
§ 3. PRINCIPAL VALUE OF THE CAUCHY TYPE INTEGRAL	7
3.1. Improper integral (7). 3.2. Principal value of singular integral (8). 3.3. Many-valued functions (10). 3.4. Principal value of singular curvilinear integral (13). 3.5. Properties of the singular integral (16).	
§ 4. LIMITING VALUES OF THE CAUCHY TYPE INTEGRAL. INTEGRALS OVER THE REAL AXIS	20
4.1. The basic lemma (20). 4.2. The Sokhotski formulae (23). 4.3. The conditions ensuring that an arbitrary complex function is the boundary value of a function analytic in the domain (25). 4.4. Limiting values of the derivatives. Derivatives of limiting values. Derivatives of a singular integral (28). 4.5. The Sokhotski formulae for corner points of a contour (31). 4.6. Integrals of the Cauchy type over the real axis (33).	
§ 5. PROPERTIES OF THE LIMITING VALUES OF THE CAUCHY TYPE INTEGRAL	38
5.1. The limiting values satisfy the Hölder condition (38). 5.2. Extension of the assumptions (41). 5.3. Some new results (43).	
§ 6. THE HILBERT FORMULAE FOR THE LIMITING VALUES OF THE REAL AND IMAGINARY PARTS OF AN ANALYTIC FUNCTION	44
6.1. The Cauchy and Schwarz kernels (44). 6.2. The Hilbert formulae (45).	
§ 7. THE CHANGE OF THE ORDER OF INTEGRATION IN A REPEATED SINGULAR INTEGRAL	46
7.1. The case when one integral is ordinary (46). 7.2. The transposition formula (49). 7.3. Inversion of the singular integral with the Cauchy kernel for the case of a closed contour (52).	

§ 8. BEHAVIOUR OF THE CAUCHY TYPE INTEGRAL AT THE ENDS OF THE CONTOUR OF INTEGRATION AND AT THE POINTS OF DENSITY DISCONTINUITIES	53
8.1. The case of density satisfying the Hölder condition on L , including the ends (53).	
8.2. The case of discontinuity of the first kind (54).	
8.3. The particular case of a power singularity (55).	
8.4. The general case of a power singularity (60).	
8.5. Singularity of logarithmic type (62).	
8.6. Singularities of power-logarithmic type (63).	
8.7. Integral of the Cauchy type over a complicated contour (64).	
§ 9. LIMITING VALUES OF GENERALIZED INTEGRALS AND DOUBLE CAUCHY INTEGRALS	66
9.1. Formulation of the problem (66).	
9.2. Formulae analogous to the Sokhotski formulae for the Cauchy type integral (67).	
9.3. The formula for the change of the order of integration (68).	
9.4. Multiple Cauchy integrals. Formulation of the problem (69).	
9.5. Singular double integral. Poincaré–Bertrand formula (70).	
9.6. Sokhotski's formulae (71).	
§ 10. INTEGRAL OF THE CAUCHY TYPE AND POTENTIALS	73
§ 11. HISTORICAL NOTES	75
PROBLEMS ON CHAPTER I	77
II. RIEMANN BOUNDARY VALUE PROBLEM	
§ 12. THE INDEX	85
12.1. Definition and basic properties (85).	
12.2. Computation of the index (87).	
§ 13. SOME AUXILIARY THEOREMS	90
§ 14. THE RIEMANN PROBLEM FOR A SIMPLY-CONNECTED DOMAIN	90
14.1. Formulation of the problem (90).	
14.2. Determination of sectionally analytic function in accordance with given jump (91).	
14.3. Solution of the homogeneous problem (92).	
14.4. The canonical function of the homogeneous problem (95).	
14.5. Solution of the non-homogeneous problem (96).	
14.6. Examples (99).	
14.7. The Riemann problem for the semi-plane (104).	
§ 15. EXCEPTIONAL CASES OF THE RIEMANN PROBLEM	107
15.1. The homogeneous problem (107).	
15.2. The non-homogeneous problem (109).	
§ 16. RIEMANN PROBLEM FOR MULTIPLY-CONNECTED DOMAIN. SOME NEW RESULTS	113
16.1. Formulation of the problem (113).	
16.2. Solution of the problem (115).	
16.3. Some new results (118).	
§ 17. RIEMANN BOUNDARY VALUE PROBLEM WITH SHIFT	121
17.1. Formulation of the problem and general remarks (121).	
17.2. Problem with zero jump (123).	
17.3. Problem with given jump (126).	
17.4. The homogeneous problem with zero index (128).	
17.5. Reducing the shift problem to the ordinary Riemann problem (129).	

§ 18. OTHER GENERALIZED PROBLEMS	133
18.1. Formulation of the problems and notation (133). 18.2. Reduction to the simplest case (135).	
§ 19. HISTORICAL NOTES	137
PROBLEMS ON CHAPTER II	138
 III. SINGULAR INTEGRAL EQUATIONS WITH CAUCHY KERNEL 	
§ 20. BASIC CONCEPTS AND NOTATION	143
20.1. Singular integral equation (143). 20.2. Fundamental results of the theory of Fredholm integral equations (146).	
§ 21. THE DOMINANT EQUATION	148
21.1. Reduction to the Riemann boundary value problem (148). 21.2. Solution of the dominant equation (150). 21.3. The solution of the equation adjoint to the dominant equation (154). 21.4. Examples (156). 21.5. Approximate solution (157). 21.6. The behaviour of the solution at corner points (159).	
§ 22. REGULARIZATION OF THE COMPLETE EQUATION	161
22.1. Product of singular operators (161). 22.2. Regularizing operator (165). 22.3. Methods of regularization (167). 22.4. Relation between solutions of singular and regularized equations (168).	
§ 23. FUNDAMENTAL PROPERTIES OF SINGULAR EQUATIONS	171
23.1. Some properties of adjoint operators (171). 23.2. Fundamental theorem on singular integral equations (Noether's theorems) (172). 23.3. Some corollaries (177).	
§ 24. EQUIVALENT REGULARIZATION. THE THIRD METHOD OF REGULARIZATION	178
24.1. Statement of the problem. Various interpretations of the concept of equivalent regularization (178). 24.2. Equivalent regularization of an operator (179). 24.3. Equivalent regularization from the left of the singular equation. The necessary condition (180). 24.4. Conjugate equation. Another form of the conditions of solubility of the non-homogeneous equation (182). 24.5. Theorem on equivalent regularization of an equation (184). 24.6. Regularization by solving the dominant equation (the method of Carleman-Vekua) (186). 24.7. Example (189).	
§ 25. EXCEPTIONAL CASES OF SINGULAR INTEGRAL EQUATIONS	194
25.1. Solution of the dominant equation (195). 25.2. Regularization of the complete equation (197).	
§ 26. HISTORICAL NOTES	199
PROBLEMS ON CHAPTER III	201

IV. HILBERT BOUNDARY VALUE PROBLEM AND SINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL

- § 27. FORMULATION OF THE HILBERT PROBLEM AND SOME AUXILIARY FORMULAE 207
 27.1. Formulation of the Hilbert problem (207). 27.2. The Schwarz operator for simply-connected domain (208). 27.3. Determination of an analytic function possessing a pole, in terms of the value of its real part on the contour (Problem A) (210).
- § 28. REGULARIZING FACTOR 213
 28.1. Definition of the regularizing factor (213). 28.2. Real regularizing factor (214). 28.3. Regularizing factor with constant modulus (217). 28.4. Other forms of the regularizing factor (218).
- § 29. THE HILBERT BOUNDARY VALUE FOR SIMPLY-CONNECTED DOMAINS 220
 29.1. The homogeneous problem (220). 29.2. The non-homogeneous problem (221). 29.3. Problem for the unit circle (223). 29.4. The Hilbert problem for exterior domain (224). 29.5. Examples (227).
- § 30. RELATION BETWEEN THE HILBERT AND RIEMANN PROBLEMS 228
 30.1. Comparison of the formulae representing the solutions of the boundary value problems (228). 30.2. Connection between the boundary conditions (233).
- § 31. SINGULAR INTEGRAL EQUATION WITH HILBERT KERNEL 236
 31.1. Connection of the dominant equation with the Hilbert boundary value problem (236). 31.2. The homogeneous equation (237). 31.3. The non-homogeneous equation (240). 31.4. Equation with constant coefficients (244). 31.5. The complete equation and its regularization (246). 31.6. Basic properties of equation with Hilbert kernel (247).
- § 32. BOUNDARY VALUE PROBLEMS FOR POLYHARMONIC AND POLYANALYTIC FUNCTIONS, REDUCIBLE TO THE HILBERT BOUNDARY VALUE PROBLEM 249
 32.1. Representation of polyharmonic and polyanalytic functions by analytic functions (249). 32.2. Formulation of the boundary value problems for polyanalytic functions (252). 32.3. Boundary value problems for a circle (254). 32.4. Boundary value problems for domains mapped onto the circle by means of rational functions (259). 32.5. Solution of the basic problem for the theory of elasticity for the domain bounded by Pascal's limaçon (261).
- § 33. THE INVERSE BOUNDARY VALUE PROBLEM FOR ANALYTIC FUNCTIONS 265
 33.1. Formulation of the problem (265). 33.2. Solution of the interior problem (266). 33.3. Other methods of prescribing boundary values (269). 33.4. Solution of the exterior problem (270). 33.5. The number of solutions of the exterior problem (273). 33.6. The Schwarz problem with a logarithmic singularity on the contour

(275). 33.7. Singular points of the contour (277). 33.8. The single-sheet nature of the solution (279). 33.9. Some further topics (281).

§ 33*. HISTORICAL NOTES	284
PROBLEMS ON CHAPTER IV	286

V. VARIOUS GENERALIZED BOUNDARY VALUE PROBLEMS

§ 34. BOUNDARY VALUE PROBLEM OF HILBERT TYPE, WITH THE BOUNDARY CONDITION CONTAINING DERIVATIVES	291
34.1. Formulation of the problem and various forms of the boundary conditions (291). 34.2. Representation of an analytic function by a Cauchy type integral with a real density (294). 34.3. The Cauchy type integral the density of which is the product of a given complex function and a real function (297). 34.4. Integral representation of an analytic function the m th derivative of which is representable by the Cauchy type integral (300). 34.5. Reduction of the boundary value problem to a Fredholm integral equation (304). 34.6. Problems of solubility of the boundary value problem (306). 34.7. Other methods of investigation (310).	
§ 35. BOUNDARY VALUE PROBLEM OF RIEMANN TYPE WITH THE BOUNDARY CONDITION CONTAINING DERIVATIVES	316
35.1. An integral representation for sectionally analytic function (316). 35.2. Solution of the boundary value problem (319). 35.3. New method of solving the problem (322). 35.4. Singular integro-differential equation (324).	
§ 36. THE HILBERT BOUNDARY VALUE PROBLEM FOR MULTIPLY-CONNECTED DOMAINS	326
36.1. The Dirichlet problem for multiply-connected domains (327). 36.2. The Schwarz operator for a multiply-connected domain (332). 36.3. Problem A for a multiply-connected domain (335). 36.4. The regularizing factor (336). 36.5. The solution of the homogeneous Hilbert problem in the class of many-valued functions (338). 36.6. Integral equation of the Hilbert problem (340). 36.7. The adjoint integral equation and the adjoint Hilbert problem (341). 36.8. Investigation of the solubility problems (344). 36.9. Investigation of the cases $\kappa = 0$ and $\kappa = m - 1$ (348). 36.10. Connection with the mapping onto a plane with slits (352). 36.11. Concluding remarks (355). 36.12. Some new results (356).	
§ 36*. INVERSE BOUNDARY VALUE PROBLEM FOR A MULTIPLY-CONNECTED DOMAIN	358
36*.1. Formulation of the problem (358). 36*.2. Solution of the interior problem (359). 36*.3. Solubility conditions (360).	
§ 37. GENERAL BOUNDARY VALUE PROBLEM OF RIEMANN TYPE FOR MULTIPLY-CONNECTED DOMAINS	362
37.1. The integral representation (362). 37.2. Boundary value problem and integro-differential equation (364).	

§ 38. BOUNDARY VALUE PROBLEMS FOR EQUATIONS OF ELLIPTIC TYPE	366
38.1. General information (366). 38.2. Classical methods of solution (367). 38.3. Integral representation of solutions (369). 38.4. The general boundary value problem (374).	
§ 39. BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF ELLIPTIC EQUATIONS	375
39.1. Various forms of the system and general remarks (375). 39.2. Functions of class C (377). 39.3. The fundamental solution (378). 39.4. The normal form of the system (380). An auxiliary representation of the solutions and its implications (382). 39.6. Integral representation of solutions (384). 39.7. Boundary value problems (386). 29.8. Riemann boundary value problem. Formulation and auxiliary relationships (390). 39.9. The solution of the Riemann boundary value problem (393). 39.10. Additional remarks (397).	
§ 40. HISTORICAL NOTES	398
PROBLEMS ON CHAPTER V	399

VI. BOUNDARY VALUE PROBLEMS AND SINGULAR INTEGRAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND OPEN CONTOURS

§ 41. SOLUTION OF THE RIEMANN PROBLEM WITH DISCONTINUOUS COEFFICIENTS BY REDUCTION TO A PROBLEM WITH CONTINUOUS COEFFICIENTS	407
41.1. Formulation of the problem and determination of the function in terms of a given jump (407). 41.2. Fundamental auxiliary functions (410). 41.3. Reduction to a problem with continuous coefficients. The simplest case (411). 41.4. Solution of the homogeneous problem (413). 41.5. Solution of the non-homogeneous problem (416).	
§ 42. RIEMANN BOUNDARY VALUE PROBLEM FOR OPEN CONTOURS	420
42.1. Formulation and solution of the problem (421). 42.2. Example (423). 42.3. Inversion of the Cauchy type integral (426).	
§ 43. DIRECT SOLUTION OF THE RIEMANN PROBLEM	428
43.1. The problem for an open contour (429). 43.2. The problem with discontinuous coefficients (430). 43.3. Examples (431). 43.4. Concluding remarks (435).	
§ 44. RIEMANN PROBLEM FOR A COMPLICATED CONTOUR	436
44.1. Formulation of the problem (436). 44.2. A new method of solution of the Riemann problem for a closed curve (437). 44.3. The general case (438). 44.4. The case of coincidence of the ends (441).	

§ 45. EXCEPTIONAL CASES AND THE GENERAL CONCEPT OF INDEX	442
45.1. Introductory remarks (442). 45.2. The homogeneous Riemann problem (443). 45.3. Exceptional points at contour ends (447).	
§ 46. HILBERT BOUNDARY VALUE PROBLEM WITH DISCONTINUOUS COEFFICIENTS	449
46.1. Hilbert problem for the semi-plane (449). 46.2. Example (451). 46.3. Mixed boundary value problem for analytic functions (454). 46.4. The Dirichlet problem and its modifications for the plane with slits (457).	
§ 47. THE DOMINANT EQUATION FOR OPEN CONTOURS	462
47.1. Basic concepts and notation (462). 47.2. Solution of the dominant equation (464). 47.3. Solution of the equation adjoint to the dominant equation (469). 47.4. Examples (471).	
§ 48. COMPLETE EQUATION FOR OPEN CONTOURS	472
48.1. Regularization by solving the dominant equation (472). 48.2. Investigation of the regularized equation (474). 48.3. Other methods of regularization. Equivalent regularization (476). 48.4. Basic properties of the singular equation (478).	
§ 49. THE GENERAL CASE	480
49.1. Equation on a complex contour and with discontinuous coefficients (480). 49.2. Example (482). 49.3. Exceptional cases (484). 49.4. Approximate methods (484).	
§ 50. HISTORICAL NOTES	485
PROBLEMS ON CHAPTER VI	487
VII. INTEGRAL EQUATIONS SOLUBLE IN CLOSED FORM	
§ 51. EQUATIONS WITH AUTOMORPHIC KERNELS AND A FINITE GROUP	495
51.1. Some results of the theory of finite groups of linear fractional transformations and of automorphic functions (496). 51.2. Reduction of a complete singular equation to a boundary value problem (499). 51.3. Solution of the boundary value problem (502). 51.4. Solution of the integral equation (505). 51.5. Example (506). 51.6. The case when the auxiliary analytic function does not vanish at infinity (509). 51.7. Additional remarks (512).	
§ 52. CONTINUATION. THE CASE OF AN INFINITE GROUP	512
52.1. The Riemann boundary value problem (513). 52.2. The singular integral equation (515). 52.3. Some applications (516). 52.4. Integral equation with a non-fundamental automorphic function (522). 52.5. Example (525).	
§ 53. SOME TYPES OF INTEGRAL EQUATIONS WITH POWER AND LOGARITHMIC KERNELS	526
53.1. Abel integral equation (527). 53.2. Integral with a power kernel (529). 53.3. The generalized Abel integral equation (531). 53.4. Integral with a logarithmic kernel (535). 53.5. Integral equations with logarithmic kernels (538). 53.6. Various possible generalizations (541).	

§ 54. HISTORICAL NOTES	544
PROBLEMS ON CHAPTER VII	546
REFERENCES	551
INDEX	559
OTHER TITLES IN THE SERIES	562