

# CONTENTS

## INTRODUCTION

SECTION	PAGE
0.1. The positive integers	3
0.2. The fundamental operations on integers	4
0.3. The rational numbers	7
0.4. The irrational numbers	9
0.5. The real number system	12
Problems	18

## I. SETS, SEQUENCES, AND FUNCTIONS

1.1. Bounds and limits of sets and sequences	20
1.2. Functions and their properties	30
1.3. Sequences of functions and uniform convergence	36
Problems	41

## II. METRIC PROPERTIES OF SETS

2.1. Notation and definitions	45
2.2. Descriptive properties of sets	46
2.3. Metric properties of sets	48
2.4. Measurability and measurable sets	54
2.5. Further descriptive properties of sets	60
2.6. Measure-preserving transformations and non-measurable sets	61
2.7. A non-measurable set	62
Problems	64

## III. THE LEBESGUE INTEGRAL

SECTION	PAGE
3.1. Measurable functions	66
3.2. The Lebesgue integral	67
3.3. The Riemann integral	69
3.4. The extension of the definition of the Lebesgue integral to unbounded functions	73
3.5. Further properties of measurable functions	76
Problems	78

IV. PROPERTIES OF THE LEBESGUE  
INTEGRAL

4.1. Notation and conventions	81
4.2. Properties of the Lebesgue integral	81
4.3. Definitions of summability and their extension to unbounded sets	88
4.4. The integrability of sequences	92
4.5. Integrals containing a parameter	95
4.6. Further theorems on sequences of functions	97
4.7. The ergodic theorem	100
Problems	106

V. METRIC DENSITY AND FUNCTIONS  
OF BOUNDED VARIATION

5.1. The Vitali covering theorem	110
5.2. Metric density of sets	114
5.3. Approximate continuity	118
5.4. Functions of bounded variation	118
5.5. Upper and lower derivatives	122
5.6. Functions of sets	125
5.7. The summability of the derivative of a function of bounded variation	127
5.8. Functions of sets	131
Problems	137

## VI. THE INVERSION OF DERIVATIVES

SECTION	PAGE
6.1. Functions defined by integrals, $F(x) = L(f,a,x)$	140
6.2. The inversion of derivatives which are not summable	146
6.3. The integrals of Denjoy and other generalized integrals	158
6.4. Descriptive definitions of generalized integrals Problems	159 162

## VII. DERIVED NUMBERS AND DERIVATIVES

7.1. Derivatives or derived numbers	165
7.2. The Weierstrass non-differentiable function	168
7.3. A function which has no unilateral derivative	172
7.4. The derived numbers of arbitrary functions defined on arbitrary sets	181
7.5. Approximate derived numbers over arbitrary sets	187
7.6. Approximate derived numbers of measurable functions, and relations between arbitrary functions and measurable functions	199

## VIII. THE STIELTJES INTEGRAL

8.1. The Riemann-Stieltjes Integral	204
8.2. Properties of the Riemann-Stieltjes integral	205
8.3. Interval functions and measure functions	211
8.4. Linear functionals	212
BIBLIOGRAPHY	221
INDEX OF SUBJECTS	225
INDEX OF AUTHORS	231