Contents

Preface *xiii*
Acknowledgments *xvii*
Basic Notation and Terminology *xix*

Part I Theory of Majorization

Chapter 1 Introduction

A. Motivation and Basic Definitions 3
B. Majorization as a Partial Ordering 12
C. Order-Preserving Functions 13
D. Various Generalizations of Majorization 14

Chapter 2 Doubly Stochastic Matrices

A. Doubly Stochastic Matrices and Permutation Matrices 18
B. Characterization of Majorization Using Doubly Stochastic Matrices 21
C. Doubly Substochastic Matrices and Weak Majorization 24
D. Doubly Superstochastic Matrices and Weak Majorization 29
E. Orderings on D 32
F. Proofs of Birkhoff's Theorem and Refinements 34
G. Classes of Doubly Stochastic Matrices 38
H. More Examples of Doubly Stochastic and Doubly Substochastic Matrices 45
I. Properties of Double Stochastic Matrices 48
J. Diagonal Equivalence of Nonnegative Matrices and Doubly Stochastic Matrices 51

Chapter 3 **Schur-Convex Functions**

A. Characterization of Schur-Convex Functions 54
B. Compositions Involving Schur-Convex Functions 60
C. Some General Classes of Schur-Convex Functions 64
D. Examples I. Sums of Convex Functions 70
E. Examples II. Products of Logarithmically Concave Functions 73
F. Examples III. Elementary Symmetric Functions 78
G. Symmetrization of Convex and Schur-Convex Functions: Muirhead's Theorem 82
H. Shur-Convex Functions on $\mathscr{D}$ and Their Extension to $\mathscr{R}^n$ 91
I. Miscellaneous Specific Examples 96
J. Integral Transformations Preserving Schur-Convexity 99

Chapter 4 **Equivalent Conditions for Majorization**

A. Characterization by Linear Transformations 107
B. Characterization in Terms of Order-Preserving Functions 108
C. A Geometric Characterization 113

Chapter 5 **Preservation and Generation of Majorization**

A. Operations Preserving Majorization 115
B. Generation of Majorization 128
C. Maximal and Minimal Vectors under Constraints 132
D. Majorization in Integers 134

Chapter 6 **Rearrangements and Majorization**

A. Majorizations from Additions of Vectors 139
B. Majorizations from Functions of Vectors 143
C. Weak Majorizations from Rearrangements 146
D. L-Superadditive Functions—Properties and Examples 150
E. Inequalities without Majorization 156
F. A Relative Arrangement Partial Order 158

Part II **Mathematical Applications**

Chapter 7 **Combinatorial Analysis**

A. Some Preliminaries on Graphs, Incidence Matrices, and Networks 171
B. Conjugate Sequences 173
C. The Theorem of Gale and Ryser 176
D. Some Applications of the Gale–Ryser Theorem 179
E. s-Graphs and a Generalization of the Gale–Ryser Theorem 183
F. Tournaments 185
G. Edge Colorings in Graphs 189

Chapter 8 **Geometric Inequalities**

A. Inequalities for the Angles of a Triangle 193
B. Inequalities for the Sides of a Triangle 198
C. Inequalities for the Exradii and Altitudes 203
D. Inequalities for the Sides, Exradii, and Medians 205
E. Isoperimetric-Type Inequalities for Plane Figures 208

Chapter 9 **Matrix Theory**

A. Notation and Preliminaries 215
B. Diagonal Elements and Characteristic Roots of a Hermitian Matrix 218
C. Characteristic Roots of a Hermitian Matrix and Its Principal Submatrices 225
D. Diagonal Elements and Singular Values 228
E. Absolute Value of Characteristic Roots and Singular Values 231
F. Characteristic Roots and Various Singular Values 237
G. Characteristic Roots and Singular Values of A, B, and $A + B$ 241
H. Characteristic Roots and Singular Values of A, B, and AB 246
I. Absolute Values of Characteristic Roots, Row Sums, and Variations of Hadamard's Inequality 253
J. Diagonal Elements and Characteristic Roots of a Totally Positive Matrix and of an M-Matrix 259

Chapter 10 **Numerical Analysis**

A. Unitarily Invariant Norms and Symmetric Gauge Functions 263
B. Matrices Closest to a Given Matrix 264
C. Condition Numbers and Linear Equations 270
D. Condition Numbers of Submatrices and Augmented Matrices 273
E. Condition Numbers and Norms 274

Part III **Stochastic Applications**

Chapter 11 **Stochastic Majorizations**

A. Introduction 281
B. Convex Functions and Exchangeable Random Variables 285
C. Families of Distributions Parameterized to Preserve Symmetry and Convexity 290
D. Some Consequences of the Stochastic Majorization $E_1(P_1)$ 295
E. Families of Distributions Parameterized to Preserve Schur-Convexity 296
F. Additional Stochastic Majorizations and Properties 311
G. Weak Stochastic Majorizations 317
H. Additional Stochastic Weak Majorizations and Properties 324

Chapter 12 **Probabilistic and Statistical Applications**

A. Sampling from a Finite Population 331
B. Majorization Using Jensen's Inequality 343
C. Probabilities of Realizing at Least k of n Events 345
D. Expected Values of Ordered Random Variables 348
E. Characteristic Roots of a Random Matrix 355
F. Special Results for Bernoulli and Geometric Random Variables 359
G. Weighted Sums of Symmetric Random Variables 361
H. Stochastic Ordering from Ordered Random Variables 366
I. Another Stochastic Majorization Based on Stochastic Ordering 370
J. Peakedness of Distributions of Linear Combinations 372
K. Tail Probabilities for Linear Combinations 375
L. Schur-Concave Distribution Functions and Survival Functions 378
M. Bivariate Probability Distributions with Fixed Marginals 381
N. Random Variables Taking Values in a Finite Group 383

Chapter 13 **Additional Statistical Applications**

A. Unbiasedness of Tests and Monotonicity of Power Functions 386
B. Comparison of Linear Estimators 392
C. Ranking and Selection 395
D. Majorization in Reliability Theory 402
E. Entropy 405
F. Measuring Inequality and Diversity 407

Part IV **Generalizations**

Chapter 14 **Orderings Extending Majorization**

A. p-Majorization 417
B. G-Majorization 422
C. Partial Orderings Induced by Convex Cones 424
D. Other Relatives of Majorization 427

Chapter 15 **Multivariate Majorization**

A. Some Definitions 429
B. Theorems Suggesting Alternative Definitions 433
C. The Order-Preserving Functions 434
D. Additional Extensions 436
E. A Probability Inequality 438

Part V Complementary Topics

Chapter 16 Convex Functions and Some Classical Inequalities

A. Monotone Functions 443
B. Convex Functions 445
C. Jensen's Inequality 454
D. Some Additional Fundamental Inequalities 457
E. Matrix-Monotone and Matrix-Convex Functions 462
F. Real-Valued Functions of Matrices 474

Chapter 17 Stochastic Ordering

A. Definitions 481
B. Some Equivalent Conditions and Basic Theorems 483
C. Stochastic Ordering in the Sense of Componentwise Ordering 485

Chapter 18 Total Positivity

A. Totally Positive Functions 487
B. Pólya Frequency Functions 492
C. Pólya Frequency Sequences 495

Chapter 19 Matrix Factorizations, Compounds, Direct Products, and *M*-Matrices

A. Characteristic Root Decompositions 496
B. Singular Value Decomposition 498
C. Square Roots and the Polar Decomposition 499
D. A Duality between Positive Semidefinite Hermitian Matrices and Complex Matrices 501
E. Simultaneous Reduction of Two Hermitian Matrices 502
F. Compound Matrices 502
G. Kronecker Products 507
H. *M*-Matrices 507

Chapter 20 Extremal Representations of Matrix Functions

A. Characteristic Roots of a Hermitian Matrix 509
B. Singular Values 514
C. Other Extremal Representations 519

Biographies 521

References 531

Author Index 553
Subject Index 559