## Contents

Introduction to Manifolds

4. Further Examples of Manifolds. Cutting and Pasting

1. Differentiability for Functions of Several Variables

2. Differentiability of Mappings and Jacobians

1. Preliminary Comments on  $\mathbb{R}^n$ 2. R<sup>n</sup> and Euclidean Space 3. Topological Manifolds

5. Abstract Manifolds. Some Examples 18

## PREFACE

T.

Notes

	, II C
3.	The Space of Tangent Vectors at a Point of $\mathbb{R}^n$ 29
4.	Another Definition of $T_a(\mathbf{R}^n)$ 32
5.	Vector Fields on Open Subsets of $\mathbb{R}^n$ 37
	The Inverse Function Theorem 41
7.	The Rank of a Mapping 46
	Notes 50
Ш	. Differentiable Manifolds and Submanifolds
1.	The Definition of a Differentiable Manifold 52
2.	Further Examples 60
3.	Differentiable Functions and Mappings 65
	Rank of a Mapping. Immersions 69
5.	Submanifolds 75
6.	Lie Groups 81
7.	The Action of a Lie Group on a Manifold. Transformation Groups 89
8.	The Action of a Discrete Group on a Manifold 95
	Covering Manifolds 100
	Notes 104

II. Functions of Several Variables and Mappings

11

20

CONTENTS viii

122

IV. Vector Fields on a Manifold

1.	The Tangent Space at a Point of a Manifold 106 Vector Fields 115	
2. 3.	Vector Fields 115 One-Parameter and Local One-Parameter Groups Acting on a Manifold	
4.	The Existence Theorem for Ordinary Differential Equations 130	
5.	Some Examples of One-Parameter Groups Acting on a Manifold 138	
6.	One-Parameter Subgroups of Lie Groups 145	
7.	The Lie Algebra of Vector Fields on a Manifold 149	
8.	Frobenius' Theorem 156	
9.	Homogeneous Spaces 164	
۸	Notes 171 pendix Partial Proof of Theorem 4.1 172	
Appendix Partial Proof of Theorem 4.1 172		
V.	Tensors and Tensor Fields on Manifolds	
1.	Tangent Covectors 175	
	Covectors on Manifolds 176	
	Covector Fields and Mappings 178	
2.	Bilinear Forms. The Riemannian Metric 181	
3.	Riemannian Manifolds as Metric Spaces 185	
4.	Partitions of Unity 191	
5.	Some Applications of the Partition of Unity 193 Tensor Fields 197	
Э.	Tensors on a Vector Space 197	
	Tensor Fields 199	
	Mappings and Covariant Tensors 200	
	The Symmetrizing and Alternating Transformations 201	
6.	Multiplication of Tensors 204	
	Multiplication of Tensors on a Vector Space 205	
	Multiplication of Tensor Fields 206	
	Exterior Multiplication of Alternating Tensors 207	
7	The Exterior Algebra on Manifolds 211 Orientation of Manifolds and the Volume Element 213	
7. 8.	Orientation of Manifolds and the Volume Element 213  Exterior Differentiation 217	
ο.	An Application to Frobenius' Theorem 221	
	Notes 225	
	11003 223	
VI. Integration on Manifolds		
1.	Integration in <b>R</b> ". Domains of Integration 227 Basic Properties of the Riemann Integral 228	
2.	A Generalization to Manifolds 233	
	Integration on Riemannian Manifolds 237	
3.	Integration on Lie Groups 241	
4.	Manifolds with Boundary 248	
5.	Stokes's Theorem for Manifolds with Boundary 256	
6.	Homotopy of Mappings. The Fundamental Group 263	
	Homotopy of Paths and Loops. The Fundamental Group 265	
7.	Some Applications of Differential Forms. The de Rham Groups 271	
	The Homotopy Operator 274	

The de Rham Groups of Lie Groups 282 9. Covering Spaces and the Fundamental Group 286 Notes 292
VII. Differentiation on Riemannian Manifolds
1. Differentiation of Vector Fields along Curves in R <sup>n</sup> 294  The Geometry of Space Curves 297  Curvature of Plane Curves 301
<ol> <li>Differentiation of Vector Fields on Submanifolds of R<sup>n</sup> 303</li> <li>Formulas for Covariant Derivatives 308</li> <li>V<sub>Y</sub> Y and Differentiation of Vector Fields 310</li> </ol>
3. Differentiation on Riemannian Manifolds 313
Constant Vector Fields and Parallel Displacement 319 4. Addenda to the Theory of Differentiation on a Manifold 321
4. Addenda to the Theory of Differentiation on a Manifold 321  The Curvature Tensor 321
The Riemannian Connection and Exterior Differential Forms
5. Geodesic Curves on Riemannian Manifolds 326
<ul><li>6. The Tangent Bundle and Exponential Mapping. Normal Coordinate</li><li>7. Some Further Properties of Geodesics 338</li></ul>
8. Symmetric Riemannian Manifolds 347
9. Some Examples 353 Notes 360
VIII. Curvature
1. The Geometry of Surfaces in E <sup>3</sup> 362  The Principal Curvatures at a Point of a Surface 366
2. The Gaussian and Mean Curvatures of a Surface 370 The Theorema Egregium of Gauss 373
3. Basic Properties of the Riemann Curvature Tensor 378
4. The Curvature Forms and the Equations of Structure 5. Differentiation of Covariant Tensor Fields 391
<ul><li>5. Differentiation of Covariant Tensor Fields 391</li><li>6. Manifolds of Constant Curvature 399</li></ul>
Spaces of Positive Curvature 402
Spaces of Zero Curvature 404
Spaces of Constant Negative Curvature 405
Notes 410
References 413
INDEX 417

8. Some Further Applications of de Rham Groups