### **CONTENTS**

#### INTRODUCTION, Page 1.

#### PART 1 THE INTEGRAL, Page 5.

- 1 THE RIEMANN INTEGRAL AND STEP FUNCTIONS, Page 7.
  - 1.1. The Riemann Integral, 7.
  - 1.2. Lower and Upper Integrals, 8.
  - 1.3. Step Functions, 11.
  - 1.4. Sets of Measure Zero and Sets of Full Measure, 13.
  - 1.5. Further Properties of Step Functions, 15.
  - 1.6. Application to the Theory of the Riemann Integral, 17.
  - \*1.7. Invariant Definition of Lower and Upper Functions. Lebesgue's Criterion for Riemann Integrability, 18.
    - Generalization of the Riemann Integral: The Key Idea, 20.
       Problems, 21.
- 2 GENERAL THEORY OF THE INTEGRAL, Page 23.
  - 2.1. Elementary Functions and the Elementary Integral, 23.

CONTENTS

- 2.2. Sets of Measure Zero and Sets of Full Measure, 24.
- 2.3. The Class  $L^+$ . Integration in  $L^+$ , 26.
- 2.4. Properties of the Integral in the Class  $L^+$ , 28.
- 2.5. The Class L. Integration in L, 29.
- 2.6. Levi's Theorem, 32.
- 2.7. Lebesgue's Theorem, 34.
- Summability of Almost-Everywhere Limits,
  36.
  - 2.8.1. Measurable functions, 36.
  - 2.8.2. Fatou's lemma, 37.
- Completeness of the Space L. The Riesz-Fischer Theorem, 38.
- 2.10. Fubini's Theorem, 40.
- 2.11. Integrals of Variable Sign, 44.
  - 2.11.1. Riesz's representation theorem, 44.
  - 2.11.2. Construction of a space of summable functions for the functional *I*, 47.
  - 2.11.3. Other representations of *I*. The canonical representation, 48.

## 3 THE LEBESGUE INTEGRAL IN *n*-SPACE, Page 50.

- 3.1. Relation between the Riemann Integral and the Lebesgue Integral, 50.
- 3.2. Improper Riemann Integrals and the Lebesgue Integral, 51.
- 3.3. Fubini's Theorem for Functions of Several Real Variables, 52.
- Continuous Functions as Elementary Functions, with the Riemann Integral as Elementary Integral, 54.
   Problems, 56.

#### PART 2 THE STIELTJES INTEGRAL, Page 59.

## 4 THE RIEMANN-STIELTJES INTEGRAL, Page 61.

- 4.1. Blocks and Sheets, 61.
- 4.2. Quasi-Volumes, 63.

- 4.3. Quasi-Length and the Generating Function, 64.
- 4.4. The Riemann-Stieltjes Integral and Its Properties, 66.
  - 4.4.1. Construction of the Riemann-Stieltjes integral, 66.
  - 4.4.2. Further properties, 68.
  - 4.4.3. The case of infinite B, 69.
  - 4.4.4. Equivalent quasi-volumes: a preview,
- Essential Convergence. The Helly Theorems,
- \*4.6. Applications to Analysis, 75.
  - 4.6.1. Herglotz's theorem, 75.
  - 4.6.2. Bernstein's theorem, 77.
  - 4.6.3. The Bochner-Khinchin theorem, 80.
- Structure of Signed Quasi-Volumes, 81.
  - 4.7.1. Representation of a signed quasivolume σ as the difference between two nonnegative quasi-volumes, 81.
  - 4.7.2. Other representations of  $\sigma$ . The canonical representation, 82.
  - 4.7.3. Formulas for the positive, negative and total variations, 83.
  - 4.7.4. The case n=1. Jordan's theorem, 84. Problems, 86.

#### THE LEBESGUE-STIELTJES INTEGRAL, Page 88.

- Definition of the Lebesgue-Stieltjes Integral, 88.
- 5.2. Examples, 89.
- The Lebesgue-Stieltjes Integral with Respect to a Signed Quasi-Volume, 93.
- The General Continuous Linear Functional on the Space  $C(\mathbf{B})$ , 94.
- Relation between the Quasi-Volumes  $\sigma$  and  $\tilde{\sigma}$ , 5.5. 96.
- 5.6. Continuous Quasi-Volumes, 99.
- 5.7. Equivalent Quasi-Volumes, 103.
  - Construction of the Lebesgue-Stieltjes Integral with Step Functions, as Elementary Functions, 105. Problems, 107.

#### PART 3 MEASURE, Page 111.

#### MEASURABLE SETS AND GENERAL MEASURE THEORY, Page 113.

- 6.1. More on Measurable Functions, 113.
- 6.2. Measurable Sets, 116.
- 6.3. Countable Additivity of Measure, 117.
- 6.4. Stone's Axioms, 119.
- 6.5. Characterization of Measurable Functions in Terms of Measure, 120.
- The Lebesgue Integral as Defined by 6.6. Lebesgue, 121.
- 6.7. Integration over a Measurable Subset, 123.
- 6.8. Measure on a Product Space, 125.
- \*6.9. The Space  $L_n$ , 126. Problems, 131.

#### CONSTRUCTIVE MEASURE THEORY, Page 134.

- 7.1. Semirings of Subsets, 134.
- The Subspace Generated by a Semiring of Summable Sets, 136.
- 7.3. Sufficient Semirings, 136.
- 7.4. Completely Sufficient Semirings, 139.
- 7.5. Outer Measure and the Measurability Criterion, 141.
- Measure Theory in *n*-Space. Examples, 143. 7.6.
- 7.7. Lebesgue Measure for n = 1. Inner Measure, 147.

#### Problems, 148.

### AXIOMATIC MEASURE THEORY, Page 150.

- Elementary, Borel and Lebesgue Measures, 8.1. 150.
- Lebesgue and Borel Extensions of an Ele-8.2. mentary Measure, 153.
- Construction of the Integral from a Lebesgue 8.3. Measure, 158.
- 8.4. Signed Borel Measures, 159.
- 8.5. Quasi-Volumes and Measure Theory, 162.

- 8.6. The Hahn Decomposition, 163.
- \*8.7. The General Continuous Linear Functional on the Space C(X), 166.
- \*8.8. The Lebesgue-Stieltjes Integral on an Infinite-Dimensional Space, 167.
  - Cylinder sets, blocks and quasivolumes. Extensions and projections, 168.
  - 8.8.2. Construction of the space  $L_{\omega}(X)$ . Kolmogorov's theorem, 171.
  - 8.8.3. Structure of ω-measurable sets and functions, 173.Problems, 178.

#### PART 4 THE DERIVATIVE, Page 181.

### 9 MEASURE AND SET FUNCTIONS, Page 183.

- Classification of Set Functions. Decomposition into Continuous and Discrete Components, 183.
- Decomposition of a Continuous Set Function into Absolutely Continuous and Singular Components. The Radon-Nikodým Theorem, 187.
- \*9.3. Some Consequences of the Radon-Nikodým Theorem, 190.
  - 9.3.1. The general continuous linear functional on the space L(X), 190.
  - 9.3.2. The general continuous linear functional on the space  $L_p(X)$ , 192.
  - Positive, Negative and Total Variations of the Sum of Two Set Functions, 194.
  - 9.5. The Case X = [a, b]. Absolutely Continuous Point Functions, 196.
  - The Lebesgue Decomposition, 199. Problems, 203.

# 10 THE DERIVATIVE OF A SET FUNCTION, Page 205.

- Preliminaries. Various Definitions of the Derivative, 205.
- 10.2. Differentiation with Respect to a Net, 208.

- Differentiation with Respect to a Vitali System. The Lebesgue-Vitali Theorem, 209.
- 10.4. Some Consequences of the Lebesgue-Vitali Theorem, 215.
  - 10.4.1. De Possel's theorem, 215.
  - 10.4.2. Lebesgue's theorem on differentiation of a function of bounded variation, 216.
- Differentiation with Respect to the Underlying σ-Ring, 220.
  Problems, 223.

BIBLIOGRAPHY, Page 227.

INDEX, Page 229.