

## CONTENTS

Preface . . . . .	11
<i>Chapter I. NUMBER. VARIABLE. FUNCTION</i>	
1. Real Numbers. Real Numbers as Points on a Number Scale . . . . .	13
2. The Absolute Value of a Real Number . . . . .	14
3. Variables and Constants . . . . .	16
4. The Range of a Variable . . . . .	16
5. Ordered Variables. Increasing and Decreasing Variables. Bounded Variables . . . . .	18
6. Function . . . . .	19
7. Ways of Representing Functions . . . . .	20
8. Basic Elementary Functions. Elementary Functions . . . . .	22
9. Algebraic Functions . . . . .	26
10. Polar Coordinate System . . . . .	28
<i>Exercises on Chapter I</i> . . . . .	30

### *Chapter II. LIMIT. CONTINUITY OF A FUNCTION*

1. The Limit of a Variable. An Infinitely Large Variable . . . . .	32
2. The Limit of a Function . . . . .	35
3. A Function that Approaches Infinity. Bounded Functions . . . . .	38
4. Infinitesimals and Their Basic Properties . . . . .	42
5. Basic Theorems on Limits . . . . .	45
6. The Limit of the Function $\frac{\sin x}{x}$ as $x \rightarrow 0$ . . . . .	50
7. The Number $e$ . . . . .	51
8. Natural Logarithms . . . . .	56
9. Continuity of Functions . . . . .	57
10. Certain Properties of Continuous Functions . . . . .	61
11. Comparing Infinitesimals . . . . .	63
<i>Exercises on Chapter II</i> . . . . .	66

### *Chapter III. DERIVATIVE AND DIFFERENTIAL*

1. Velocity of Motion . . . . .	69
2. Definition of Derivative . . . . .	71
3. Geometric Meaning of the Derivative . . . . .	73
4. Differentiability of Functions . . . . .	74
5. Finding the Derivatives of Elementary Functions. The Derivative of the Function $y = x^n$ , Where $n$ Is Positive and Integral . . . . .	76
6. Derivatives of the Functions $y = \sin x$ ; $y = \cos x$ . . . . .	78
7. Derivatives of: a Constant, the Product of a Constant by a Function, a Sum, a Product, and a Quotient . . . . .	80
8. The Derivative of a Logarithmic Function . . . . .	84
9. The Derivative of a Composite Function . . . . .	85
10. Derivatives of the Functions $y = \tan x$ ; $y = \cot x$ ; $y = \ln  x $ . . . . .	88

11. An Implicit Function and Its Differentiation . . . . .	89
12. Derivatives of a Power Function for an Arbitrary Real Exponent, of an Exponential Function, and a Composite Exponential Function . . . . .	91
13. An Inverse Function and Its Differentiation . . . . .	94
14. Inverse Trigonometric Functions and Their Differentiation . . . . .	98
15. Table of Basic Differentiation Formulas . . . . .	102
16. Parametric Representation of a Function . . . . .	103
17. The Equations of Certain Curves in Parametric Form . . . . .	105
18. The Derivative of a Function Represented Parametrically . . . . .	108
19. Hyperbolic Functions . . . . .	110
20. The Differential . . . . .	113
21. The Geometric Significance of the Differential . . . . .	117
22. Derivatives of Different Orders . . . . .	118
23. Differentials of Various Orders . . . . .	121
24. Different-Order Derivatives of Implicit Functions and of Functions Represented Parametrically . . . . .	122
25. The Mechanical Significance of the Second Derivative . . . . .	124
26. The Equations of a Tangent and of a Normal. The Lengths of the Subtangent and the Subnormal . . . . .	126
27. The Geometric Significance of the Derivative of the Radius Vector with Respect to the Polar Angle . . . . .	129
<i>Exercises on Chapter III</i> . . . . .	130

***Chapter IV. SOME THEOREMS ON DIFFERENTIABLE FUNCTIONS***

1. A Theorem on the Roots of a Derivative (Rolle's Theorem) . . . . .	140
2. A Theorem on Finite Increments (Lagrange's Theorem) . . . . .	142
3. A Theorem on the Ratio of the Increments of Two Functions (Cauchy's Theorem) . . . . .	143
4. The Limit of a Ratio of Two Infinitesimals (Evaluation of Indeter- minate Forms of the Type $\frac{0}{0}$ ) . . . . .	144
5. The Limit of a Ratio of Two Infinitely Large Quantities (Evaluation of Indeterminate Forms of the Type $\frac{\infty}{\infty}$ ) . . . . .	147
6. Taylor's Formula . . . . .	152
7. Expansion of the Functions $e^x$ , $\sin x$ , and $\cos x$ in a Taylor Series .	156
<i>Exercises on Chapter IV</i> . . . . .	159

***Chapter V. INVESTIGATING THE BEHAVIOUR OF FUNCTIONS***

1. Statement of the Problem . . . . .	162
2. Increase and Decrease of a Function . . . . .	163
3. Maxima and Minima of Functions . . . . .	164
4. Testing a Differentiable Function for Maximum and Minimum with a First Derivative . . . . .	171
5. Testing a Function for Maximum and Minimum with a Second Derivative . . . . .	174
6. Maxima and Minima of a Function on an Interval . . . . .	178

---

7. Applying the Theory of Maxima and Minima of Functions to the Solution of Problems . . . . .	179
8. Testing a Function for Maximum and Minimum by Means of Taylor's Formula . . . . .	181
9. Convexity and Concavity of a Curve. Points of Inflection . . . . .	183
10. Asymptotes . . . . .	189
11. General Plan for Investigating Functions and Constructing Graphs . . . . .	194
12. Investigating Curves Represented Parametrically . . . . .	199
<i>Exercises on Chapter V</i> . . . . .	203

**Chapter VI. THE CURVATURE OF A CURVE**

1. The Length of an Arc and Its Derivative . . . . .	208
2. Curvature . . . . .	210
3. Calculation of Curvature . . . . .	212
4. Calculation of the Curvature of a Line Represented Parametrically . .	215
5. Calculation of the Curvature of a Line Given by an Equation of Polar Coordinates . . . . .	215
6. The Radius and Circle of Curvature. Centre of Curvature. Evolute and Involute . . . . .	217
7. The Properties of an Evolute . . . . .	221
8. Approximating the Real Roots of an Equation . . . . .	225
<i>Exercises on Chapter VI</i> . . . . .	229

**Chapter VII. COMPLEX NUMBERS. POLYNOMIALS**

1. Complex Numbers. Basic Definitions . . . . .	233
2. Basic Operations on Complex Numbers . . . . .	234
3. Powers and Roots of Complex Numbers . . . . .	237
4. Exponential Function with Complex Exponent and Its Properties . .	240
5. Euler's Formula. The Exponential Form of a Complex Number . .	243
6. Factoring a Polynomial . . . . .	244
7. The Multiple Roots of a Polynomial . . . . .	247
8. Factorisation of a Polynomial in the Case of Complex Roots . . .	248
9. Interpolation. Lagrange's Interpolation Formula . . . . .	250
10. On the Best Approximation of Functions by Polynomials. Chebyshev's Theory . . . . .	252
<i>Exercises on Chapter VII</i> . . . . .	253

**Chapter VIII. FUNCTIONS OF SEVERAL VARIABLES**

1. Definition of a Function of Several Variables . . . . .	255
2. Geometric Representation of a Function of Two Variables . . . . .	258
3. Partial and Total Increment of a Function . . . . .	259
4. Continuity of a Function of Several Variables . . . . .	260
5. Partial Derivatives of a Function of Several Variables . . . . .	263
6. The Geometric Interpretation of the Partial Derivatives of a Function of Two Variables . . . . .	264
7. Total Increment and Total Differentials . . . . .	265
8. Approximation by Total Differentials . . . . .	268
9. Error Approximation by Differentials . . . . .	270
10. The Derivative of a Composite Function. The Total Derivative . .	273
11. The Derivative of a Function Defined Implicitly . . . . .	276
12. Partial Derivatives of Different Orders . . . . .	279

13. Level Surfaces . . . . .	283
14. Directional Derivatives . . . . .	284
15. Gradient . . . . .	286
16. Taylor's Formula for a Function of Two Variables . . . . .	290
17. Maximum and Minimum of a Function of Several Variables . . . . .	292
18. Maximum and Minimum of a Function of Several Variables Related by Given Equations (Conditional Maxima and Minima) . . . . .	300
19. Singular Points of a Curve . . . . .	305
<i>Exercises on Chapter VIII</i> . . . . .	310

***Chapter IX. APPLICATIONS OF DIFFERENTIAL CALCULUS TO SOLID GEOMETRY***

1. The Equations of a Curve in Space . . . . .	314
2. The Limit and Derivative of the Vector Function of a Scalar Argument. The Equation of a Tangent to a Curve. The Equation of a Normal Plane . . . . .	317
3. Rules for Differentiating Vectors (Vector Functions) . . . . .	322
4. The First and Second Derivatives of a Vector with Respect to the Arc Length. The Curvature of a Curve. The Principal Normal . . . . .	324
5. Osculating Plane. Binormal. Torsion . . . . .	331
6. A Tangent Plane and Normal to a Surface . . . . .	336

<i>Exercises on Chapter IX</i> . . . . .	340
--	-----

***Chapter X. INDEFINITE INTEGRALS***

1. Antiderivative and the Indefinite Integral . . . . .	342
2. Table of Integrals . . . . .	344
3. Some Properties of an Indefinite Integral . . . . .	346
4. Integration by Substitution (Change of Variable) . . . . .	348
5. Integrals of Functions Containing a Quadratic Trinomial . . . . .	351
6. Integration by Parts . . . . .	354
7. Rational Fractions. Partial Rational Fractions and Their Integration . . . . .	357
8. Decomposition of a Rational Fraction into Partial Fractions . . . . .	361
9. Integration of Rational Fractions . . . . .	365
10. Ostrogradsky's Method . . . . .	368
11. Integrals of Irrational Functions . . . . .	371
12. Integrals of the Form $\int R(x, \sqrt{ax^2 + bx + c}) dx$ . . . . .	372
13. Integration of Binomial Differentials . . . . .	375
14. Integration of Certain Classes of Trigonometric Functions . . . . .	378
15. Integration of Certain Irrational Functions by Means of Trigonometric Substitutions . . . . .	383
16. Functions Whose Integrals Cannot Be Expressed in Terms of Elementary Functions . . . . .	385

<i>Exercises on Chapter X</i> . . . . .	386
---	-----

***Chapter XI. THE DEFINITE INTEGRAL***

1. Statement of the Problem. The Lower and Upper Integral Sums . . . . .	396
2. The Definite Integral . . . . .	398
3. Basic Properties of the Definite Integral . . . . .	404
4. Evaluating a Definite Integral. Newton-Leibniz Formula . . . . .	407
5. Changing the Variable in the Definite Integral . . . . .	412

6. Integration by Parts . . . . .	413
7. Improper Integrals . . . . .	416
8. Approximating Definite Integrals . . . . .	424
9. Chebyshev's Formula . . . . .	430
10. Integrals Dependent on a Parameter . . . . .	435
<i>Exercises on Chapter XI</i> . . . . .	438

**Chapter XII. GEOMETRIC AND MECHANICAL APPLICATIONS OF THE DEFINITE INTEGRAL**

1. Computing Areas in Rectangular Coordinates . . . . .	442
2. The Area of a Curvilinear Sector in Polar Coordinates . . . . .	445
3. The Arc Length of a Curve . . . . .	447
4. Computing the Volume of a Solid from the Areas of Parallel Sections (Volumes by Slicing) . . . . .	453
5. The Volume of a Solid of Revolution . . . . .	455
6. The Surface of a Solid of Revolution . . . . .	455
7. Computing Work by the Definite Integral . . . . .	457
8. Coordinates of the Centre of Gravity . . . . .	459
<i>Exercises on Chapter XII</i> . . . . .	462

**Chapter XIII. DIFFERENTIAL EQUATIONS**

1. Statement of the Problem. The Equation of Motion of a Body with Resistance of the Medium Proportional to the Velocity. The Equation of a Catenary . . . . .	469
2. Definitions . . . . .	472
3. First-Order Differential Equations (General Notions) . . . . .	473
4. Equations with Separated and Separable Variables. The Problem of the Disintegration of Radium . . . . .	478
5. Homogeneous First-Order Equations . . . . .	482
6. Equations Reducible to Homogeneous Equations . . . . .	484
7. First-Order Linear Equations . . . . .	487
8. Bernoulli's Equation . . . . .	490
9. Exact Differential Equations . . . . .	492
10. Integrating Factor . . . . .	495
11. The Envelope of a Family of Curves . . . . .	497
12. Singular Solutions of a First-Order Differential Equation . . . . .	504
13. Clairaut's Equation . . . . .	505
14. Lagrange's Equation . . . . .	507
15. Orthogonal and Isogonal Trajectories . . . . .	509
16. Higher-Order Differential Equations (Fundamentals) . . . . .	514
17. An Equation of the Form $y^{(n)}=f(x)$ . . . . .	516
18. Some Types of Second-Order Differential Equations Reducible to First-Order Equations . . . . .	518
19. Graphical Method of Integrating Second-Order Differential Equations . . . . .	527
20. Homogeneous Linear Equations. Definitions and General Properties . . . . .	528
21. Second-Order Homogeneous Linear Equations with Constant Coefficients . . . . .	535
22. Homogeneous Linear Equations of the $n$ th Order with Constant Coefficients . . . . .	539
23. Nonhomogeneous Second-Order Linear Equations . . . . .	541
24. Nonhomogeneous Second-Order Linear Equations with Constant Coefficients . . . . .	545

---

25. Higher-Order Nonhomogeneous Linear Equations . . . . .	551
26. The Differential Equation of Mechanical Vibrations . . . . .	555
27. Free Oscillations . . . . .	557
28. Forced Oscillations . . . . .	559
29. Systems of Ordinary Differential Equations . . . . .	563
30. Systems of Linear Differential Equations with Constant Coefficients . . . . .	569
31. On Lyapunov's Theory of Stability . . . . .	576
32. Euler's Method of Approximate Solution of First-Order Differential Equations . . . . .	581
33. A Difference Method for Approximate Solution of Differential Equations Based on Taylor's Formula. Adams Method . . . . .	584
34. An Approximate Method for Integrating Systems of First-Order Differential Equations . . . . .	591
<i>Exercises on Chapter XIII</i> . . . . .	595

***Chapter XIV. MULTIPLE INTEGRALS***

1. Double Integrals . . . . .	608
2. Calculating Double Integrals . . . . .	610
3. Calculating Double Integrals (Continued) . . . . .	617
4. Calculating Areas and Volumes by Means of Double Integrals . . . . .	623
5. The Double Integral in Polar Coordinates . . . . .	626
6. Changing Variables in a Double Integral (General Case) . . . . .	633
7. Computing the Area of a Surface . . . . .	638
8. The Density of Distribution of Matter and the Double Integral . . . . .	642
9. The Moment of Inertia of the Area of a Plane Figure . . . . .	643
10. The Coordinates of the Centre of Gravity of the Area of a Plane Figure . . . . .	648
11. Triple Integrals . . . . .	650
12. Evaluating a Triple Integral . . . . .	651
13. Change of Variables in a Triple Integral . . . . .	656
14. The Moment of Inertia and the Coordinates of the Centre of Gravity of a Solid . . . . .	650
15. Computing Integrals Dependent on a Parameter . . . . .	662
<i>Exercises on Chapter XIV</i> . . . . .	663

***Chapter XV. LINE INTEGRALS AND SURFACE INTEGRALS***

1. Line Integrals . . . . .	670
2. Evaluating a Line Integral . . . . .	673
3. Green's Formula . . . . .	679
4. Conditions for a Line Integral Being Independent of the Path of Integration . . . . .	681
5. Surface Integrals . . . . .	687
6. Evaluating Surface Integrals . . . . .	689
7. Stokes' Formula . . . . .	692
8. Ostrogradsky's Formula . . . . .	697
9. The Hamiltonian Operator and Certain Applications of It . . . . .	700
<i>Exercises on Chapter XV</i> . . . . .	703

***Chapter XVI. SERIES***

1. Series. Sum of a Series . . . . .	710
2. Necessary Condition for Convergence of a Series . . . . .	713
3. Comparing Series with Positive Terms . . . . .	716

---

4. D'Alembert's Test . . . . .	718
5. Cauchy's Test . . . . .	721
6. The Integral Test for Convergence of a Series . . . . .	723
7. Alternating Series. Leibniz' Theorem . . . . .	727
8. Plus-and-Minus Series. Absolute and Conditional Convergence . . . . .	729
9. Functional Series . . . . .	733
10. Majorised Series . . . . .	734
11. The Continuity of the Sum of a Series . . . . .	736
12. Integration and Differentiation of Series . . . . .	739
13. Power Series. Interval of Convergence . . . . .	742
14. Differentiation of Power Series . . . . .	747
15. Series in Powers of $x - a$ . . . . .	748
16. Taylor's Series and Maclaurin's Series . . . . .	750
17. Examples of Expansion of Functions in Series . . . . .	751
18. Euler's Formula . . . . .	753
19. The Binomial Series . . . . .	754
20. Expansion of the Function in $(1+x)$ in a Power Series. Computing Logarithms . . . . .	756
21. Integration by Use of Series (Calculating Definite Integrals) . . . . .	758
22. Integrating Differential Equations by Means of Series . . . . .	760
23. Bessel's Equation . . . . .	763
<i>Exercises on Chapter XVI</i> . . . . .	768

*Chapter XVII. FOURIER SERIES*

1. Definition. Statement of the Problem . . . . .	776
2. Expansions of Functions in Fourier Series . . . . .	780
3. A Remark on the Expansion of a Periodic Function in a Fourier Series . . . . .	785
4. Fourier Series for Even and Odd Functions . . . . .	787
5. The Fourier Series for a Function with Period $2l$ . . . . .	789
6. On the Expansion of a Nonperiodic Function in a Fourier Series . . . . .	791
7. Approximation by a Trigonometric Polynomial of a Function Represented in the Mean . . . . .	792
8. The Dirichlet Integral . . . . .	798
9. The Convergence of a Fourier Series at a Given Point . . . . .	801
10. Certain Sufficient Conditions for the Convergence of a Fourier Series . . . . .	802
11. Practical Harmonic Analysis . . . . .	805
12. Fourier Integral . . . . .	810
13. The Fourier Integral in Complex Form . . . . .	810
<i>Exercises on Chapter XVII</i> . . . . .	812

*Chapter XVIII. EQUATIONS OF MATHEMATICAL PHYSICS*

1. Basic Types of Equations of Mathematical Physics . . . . .	815
2. Derivation of the Equation of Oscillations of a String. Formulation of the Boundary-Value Problem. Derivation of Equations of Electric Oscillations in Wires . . . . .	816
3. Solution of the Equation of Oscillations of a String by the Method of Separation of Variables (The Fourier Method) . . . . .	820
4. The Equation for Propagation of Heat in a Rod. Formulation of the Boundary-Value Problem . . . . .	823
5. Heat Propagation in Space . . . . .	825
6. Solution of the First Boundary-Value Problem for the Heat-Conductivity Equation by the Method of Finite Differences . . . . .	829

---

7. Propagation of Heat in an Unbounded Rod . . . . .	831
8. Problems That Reduce to Investigating Solutions of the Laplace Equation. Stating Boundary-Value Problems . . . . .	836
9. The Laplace Equation in Cylindrical Coordinates. Solution of the Dirichlet Problem for a Ring with Constant Values of the Desired Function on the Inner and Outer Circumferences . . . . .	841
10. The Solution of Dirichlet's Problem for a Circle . . . . .	843
11. Solution of the Dirichlet Problem by the Method of Finite Differences . . . . .	847
<i>Exercises on Chapter XVIII</i> . . . . .	850

***Chapter XIX. OPERATIONAL CALCULUS AND CERTAIN OF ITS APPLICATIONS***

1. The Initial Function and Its Transform . . . . .	854
2. Transforms of the Functions $\sigma_0(t)$ , $\sin t$ , $\cos t$ . . . . .	855
3. The Transform of a Function with Changed Scale of the Independent Variable. Transforms of the Functions $\sin at$ , $\cos at$ . . . . .	856
4. The Linearity Property of a Transform . . . . .	857
5. The Shift Theorem . . . . .	858
6. Transforms of the Functions $e^{-at}$ , $\sinh at$ , $\cosh at$ , $e^{-at} \sin at$ , $e^{-at} \cos at$ . . . . .	858
7. Differentiation of Transforms . . . . .	860
8. The Transforms of Derivatives . . . . .	861
9. Table of Transforms . . . . .	862
10. An Auxiliary Equation for a Given Differential Equation . . . . .	864
11. Decomposition Theorem . . . . .	867
12. Examples of Solutions of Differential Equations and Systems of Differential Equations by the Operational Method . . . . .	869
13. The Convolution Theorem . . . . .	871
14. The Differential Equations of Mechanical Oscillations. The Differential Equations of Electric-Circuit Theory . . . . .	873
15. Solution of the Differential Oscillation Equation . . . . .	874
16. Investigating Free Oscillations . . . . .	875
17. Investigating Mechanical and Electrical Oscillations in the Case of a Periodic External Force . . . . .	876
18. Solving the Oscillation Equation in the Case of Resonance . . . . .	878
19. The Delay Theorem . . . . .	879
<i>Exercises on Chapter XIX</i> . . . . .	880

***Subject Index***