

CONTENTS

PART THREE

THE BETTI GROUPS

Chapter

VII. CHAINS. THE OPERATOR Δ

1.	Orientation.....	2
1.1.	Orientation of the space R^n	2
1.2.	Orientation of a simplex and of a skeleton.....	3
1.3.	The body of an oriented simplex.....	5
1.4.	Extension of an orientation t^n to an orientation R^n . The product orientations $t^n R^n$ and $t_1^n t_2^n$	5
1.5.	The orientation $(e_0 t^{n-1})$	6
1.6.	Affine images of orientations.....	7
2.	Intersection Number of Planes and Simplexes.....	8
2.1.	Intersection number of planes.....	8
2.2.	Intersection number of simplexes.....	10
2.3.	Intersections and simplicial mappings.....	11
3.	Incidence Numbers.....	11
3.1.	Definition of the incidence numbers.....	11
3.2.	Properties of the incidence numbers.....	12
4.	Cell Complexes; a -complexes.....	13
4.1.	Definition of a -complexes and cell complexes.....	14
4.2.	The incidence matrices of a cell complex.....	16
5.	Chains.....	18
5.1.	Definition of a chain.....	18
5.2.	Some remarks on chains.....	20
5.3.	Monomial chains. Chains as linear forms.....	21
5.4.	Chains of a simplicial complex.....	22
5.5.	The scalar product of chains.....	23
5.6.	Extension of chains; restriction of chains to a subcomplex. The operators \mathfrak{A}_0 and E_σ	23
6.	The Lower Boundary Operator (The Operator Δ).....	24
6.1.	Definition of the Δ -boundary.....	24
6.2.	Examples of chains and their boundaries.....	26
6.3.	Cycles; chains homologous to zero; the groups $Z(\mathfrak{R})$ and $H'(\mathfrak{R})$	29
6.4.	Homologies. The symbol \sim . Linear independence of chains with respect to homology.....	30

Chapter

6.5. Restricted chains and cycles	31
6.6. Extension of chains and cycles	32
7. The Fundamental Case: \mathfrak{K} is an a -complex	34
7.1. The fundamental formula $\Delta\Delta x^r = 0$	34
7.2. Closed and open subcomplexes of an a -complex	35
7.3. Weak homology of integral cycles; the dual coefficient domain	35
8. Simplicial Images of Chains	36
8.1. Simplicial images of oriented simplexes	36
8.2. The homomorphism S_α^β of the group $L'(K_\beta)$ into the group $L'(K_\alpha)$ induced by a simplicial mapping S_α^β of a complex K_β into a complex K_α	37
8.3. Commutativity of the operators Δ and S_α^β	38
8.4. The case of open subcomplexes	39
9. Auxiliary Constructions	40
9.1. Cone over a chain	40
9.2. Application of the constructions of 9.1	41
9.3. Prism over a chain	43
9.4. Application to simplicial mappings	45
Addendum. The a -complex of the Oriented Elements of a Polyhedral Complex	47

VIII. Δ -GROUPS OF COMPLEXES (LOWER BETTI OR HOMOLOGY GROUPS)

1. Definitions. Examples. Simplest General Properties	50
1.1. Definition of the group $\Delta^r(\mathfrak{K}, \mathfrak{A})$	50
1.2. The groups $\Delta^n(\mathfrak{K}^n, \mathfrak{A})$	50
1.3. The groups $\Delta^0(K, \mathfrak{A})$	50
1.4. Simplest examples of the groups Δ^r	53
1.5. Some elementary n -complexes and their Betti groups	61
1.6. The group $\Delta^{00}(K, \mathfrak{A})$	65
1.7. Decomposition of the group $\Delta^r(\mathfrak{K}, \mathfrak{A})$ into a direct sum over the components of the complex \mathfrak{K}	66
1.8. The homomorphism of the group $\Delta^r(K_\beta, \mathfrak{A})$ into $\Delta^r(K_\alpha, \mathfrak{A})$ induced by a simplicial mapping S_α^β of a simplicial complex K_β into a simplicial complex K_α	67
2. The Groups $\Delta_0^r(\mathfrak{K})$	68
2.1. The torsion groups	68
2.2. The groups $\Delta_{00}^r(\mathfrak{K})$	69
2.3. Finite a -complexes. Homology bases	70
2.4. The Euler-Poincaré formula for a finite n -dimensional a -complex	71

Chapter

3.	Pseudomanifolds	72
3.1.	Pseudomanifolds	72
3.2.	Orientable pseudomanifolds	74
3.3.	The groups $\Delta_m^n(K^n)$ of a nonorientable n -dimensional pseudomanifold. Disorienting sequences	77
4.	Addenda and Examples	79
4.1.	The Betti groups of the complexes $ T^n $ and $\dot{T}^n = T^n \setminus T^n$	79
4.2.	Surfaces	80
4.3.	Simple pseudomanifolds. Elementary triangulations	81
4.4.	Applications to projective spaces	83
5.	Simplicial mappings of pseudomanifolds	86
5.1.	The degree of a mapping	86
5.2.	The original definition of the degree of a simplicial mapping	86
IX.	THE OPERATOR ∇ AND THE GROUPS $\nabla^r(\mathfrak{K}, \mathfrak{A})$. CANONICAL BASES. CALCULATION OF THE GROUPS $\Delta^r(\mathfrak{K}, \mathfrak{A})$ AND $\nabla^r(\mathfrak{K}, \mathfrak{A})$ BY MEANS OF THE GROUPS $\Delta_0^r(\mathfrak{K})$	
1.	The Operator ∇	90
1.1.	Definition of the chain ∇x^r	90
1.2.	The chain ∇x^r as a linear form	92
1.3.	Duality of the operators Δ and ∇	92
1.4.	The groups $Z_\nabla^r(\mathfrak{K}, \mathfrak{A})$, $H_\nabla^r(\mathfrak{K}, \mathfrak{A})$, $\nabla^r(\mathfrak{K}, \mathfrak{A})$	93
1.5.	Chains restricted to a subcomplex	94
1.6.	The groups $\nabla^0(\mathfrak{K}, \mathfrak{A}) = Z_\nabla^0(\mathfrak{K}, \mathfrak{A})$	94
1.7.	The groups $\nabla^n(K^n, J)$ of n -dimensional pseudomanifolds	95
2.	Bases of the Modules $L_0^r(\mathfrak{K})$	97
2.1.	Preliminary remarks	97
2.2.	Dual bases of $L_0^r(\mathfrak{K})$	98
2.3.	The elements of the group $L^r(\mathfrak{K}, \mathfrak{A})$ expressed in terms of a basis of the module $L_0^r(\mathfrak{K})$	99
3.	Canonical Systems of Bases. The Groups $\nabla_0^r(\mathfrak{K})$	100
3.1.	Preliminary remarks	100
3.2.	Canonical bases of the groups Z_Δ^r	100
3.3.	Canonical homology bases	101
3.4.	A system of canonical bases of the groups L^r	102
3.5.	A system of ∇ -bases for \mathfrak{K} ; the groups $\nabla_0^r(\mathfrak{K})$	105
4.	Calculation of the Groups $\Delta^r(\mathfrak{K}, \mathfrak{A})$ and $\nabla^r(\mathfrak{K}, \mathfrak{A})$ By Means of the Groups $\Delta_0^r(\mathfrak{K}, \mathfrak{A})$	109

Chapter

4.1. Calculation of the groups $\Delta'(\mathfrak{K}, \mathfrak{A})$	109
4.2. Calculation of the groups $\nabla'(\mathfrak{K}, \mathfrak{A})$	111
4.3. The coefficient domains $J, \mathfrak{R}, \mathfrak{R}_1$	112
4.4. The groups $\Delta_m'(\mathfrak{K})$ and $\nabla_m'(\mathfrak{K})$	113
4.5. Integral chains and homologies (mod m)	114
5. Calculation of the Groups $\Delta'(\mathfrak{K}, \mathfrak{A})$ and $\nabla'(\mathfrak{K}, \mathfrak{A})$ By Means of the Groups $\Delta'(\mathfrak{K}, \mathfrak{R}_1)$ and $\Delta_m'(\mathfrak{K})$	117
5.1.	117
5.2.	118
6. The Homomorphism \bar{S}_β^α of $L'(K_\alpha, \mathfrak{A})$ Into $L'(K_\beta, \mathfrak{A})$ Induced by a Simplicial Mapping S_α^β of a Complex K_β Into a Complex K_α	119
6.1. Definition of the homomorphism \bar{S}_β^α	119
6.2. The commutativity of the operators ∇ and \bar{S}_β^α	120
6.3.	120

X. INVARIANCE OF THE BETTI GROUPS

1. Formulation of the Invariance Theorems	125
1.1. Definition of the numbers $b'(\Phi)$	125
1.2. Definition of the groups $\mathfrak{B}'(\Phi)$	126
1.3. Formulation of the invariance theorem for the Betti numbers and groups	126
2. Subdivisions of Chains. Fundamental Systems of Subcomplexes and Chains. Invariance of the Δ - and ∇ -groups under Elementary and Barycentric Subdivisions	127
2.1. The isomorphism s_β^α	127
2.2. Fundamental systems of subcomplexes of a complex K .	129
2.3. Fundamental systems of chains	131
2.4. The a -complex defined by a given fundamental system of chains	135
2.5. The isomorphism β of $L'(\mathfrak{K})$ into $L'(\mathfrak{K}_\beta)$	137
2.6. The invariance of the Betti groups under elementary and barycentric subdivisions of K	139
3. Normal and Canonical Displacements in Polyhedra	139
3.1. Normal displacements of subdivisions of triangulations	139
3.2. Examples of normal homomorphisms S_α^β and \bar{S}_β^α	142
4. Canonical Systems of Bases for Subdivisions K_β of a triangulation K_α . The Homomorphism \bar{S}_β^α Dual to a Normal Homomorphism S_α^β	144
4.1. A canonical system of bases for K_β	144

Chapter

4.2. Normal homomorphisms in canonical bases.....	146
4.3. The homomorphism dual to a normal homomorphism..	147
5. Complexes $K(R, \epsilon)$. Small Displacements in Polyhedra and Compacta. The Pflastersatz and the Invariance of the Betti Numbers.....	148
5.1. The complex $K(R, \epsilon)$; ϵ -chains of a metric space R	148
5.2. ϵ -displacements.....	149
5.3. Canonical displacements.....	149
5.4. The numbers $\eta(K)$. Canonical displacements in poly- hedrala.....	150
5.5. The Pflastersatz. Invariance of the Betti numbers.....	151
6. Invariance of the Betti Groups.....	153
6.1.	153
6.2. Invariance of the Betti groups for polyhedral complexes	154
7. Invariance of Pseudomanifolds.....	155
7.1. Formulation of the theorems.....	155
7.2. Proof of Theorem 7.14.....	156

XI. THE Δ -GROUPS OF COMPACTA

1. Definition of the groups $\Delta^r(\Phi, \mathfrak{A})$	158
1.1. Proper cycles.....	158
2. Lemmas on ϵ -displacements and ϵ -homologies.....	159
2.1. Prisms and ϵ -displacements.....	159
2.2. The case of a polyhedron $\Phi = \ K_\alpha\ $	161
3. The Homomorphism of the Groups $\Delta^r(\Phi)$ Induced by a Continuous Mapping of a Compactum.....	163
3.1. The continuous image of a proper cycle.....	163
3.2.	164
3.3. Homology classification of mappings.....	164
3.4. Deformation of a continuous mapping of a proper cycle. Deformation of a proper cycle.....	166
4. The Fundamental Theorem on the Δ -groups of Polyhedra.	166
4.1. Fundamental Theorem 4.1.....	166
4.2. Construction of the homomorphism S_α^Φ of Δ_Φ^r into Δ_α^r	166
4.3. The mapping S_α^Φ is a mapping onto Δ_α^r	167
4.4. The homomorphism S_α^Φ of Δ_Φ^r onto Δ_α^r is an isomor- phism.....	168
4.5. Rules for finding the images of the isomorphisms S_α^Φ and $(S_\alpha^\Phi)^{-1}$	168
4.6. Cycles $z_\alpha^r \in Z_\alpha^r$ and homologies in $\Phi = \ K_\alpha\ $	169

Chapter

4.7. The image of a cycle $z_\alpha^r \in Z_\alpha^r$ under a continuous mapping C of a polyhedron $\Phi = \ K_\alpha\ $ into a compactum Φ' . Parametric representation and deformation of singular cycles.....	170
4.8. Orientability and orientation of closed pseudomanifolds.....	170
4.9. The homomorphism C_σ^α of $\Delta_\alpha^r = \Delta^r(K_\alpha, \mathfrak{A})$ into $\Delta_\sigma^r = \Delta^r(M_\sigma, \mathfrak{A})$ induced by a continuous mapping C_Ψ^Φ of a polyhedron $\Phi = \ K_\alpha\ $ into a polyhedron $\Psi = \ M_\sigma\ $	171
5. Simplicial Approximations to Continuous Mappings of a Polyhedron Into a Polyhedron	172
5.1. Definition of a simplicial approximation to a continuous mapping C_Ψ^Φ of $\Phi = \ K_\alpha\ $ into $\Psi = \ M_\sigma\ $	172
5.2. Fundamental property of the mapping $\tilde{S}_\sigma^{\alpha h}$	173
6. Degree of a Continuous Mapping of Pseudomanifolds.....	174
6.1. Definition of the degree.....	174
6.2. Definition of the degree of a continuous mapping of an n -cycle into an n -dimensional orientable pseudomanifold.....	174
6.3. Calculation of the degree of a mapping.....	175
6.4. Fundamental properties of the degree of a mapping	176

XII. RELATIVE CYCLES AND THEIR APPLICATIONS

1. The Complex $K(\Gamma, \epsilon)$	178
1.1. Definition of $K(\Gamma, \epsilon)$ and basic notation.....	178
1.2. Cycles and homologies in $K(\Gamma, \epsilon)$	178
1.3. (ϵ, Ψ) -displacements	180
1.4. Canonical displacements	181
2. Γ -cycles (Relative Cycles) and Γ -homologies in Φ ; the Groups $Z_\Phi^r(\Gamma, \mathfrak{A}), H_\Phi^r(\Gamma, \mathfrak{A}), \Delta_\Phi^r(\Gamma, \mathfrak{A})$	182
2.1. Definitions	182
2.2. The groups $Z_\Phi^r(\Gamma, \mathfrak{A}), H_\Phi^r(\Gamma, \mathfrak{A}), \Delta_\Phi^r(\Gamma, \mathfrak{A})$	184
2.3. Canonical and infinitesimal displacements. Isomorphism of the groups $\Delta_{\Phi_0}^r(\Gamma, \mathfrak{A})$ and $\Delta_\Phi^r(\Gamma, \mathfrak{A})$, $\Gamma \subseteq \Phi_0 \subseteq \Phi$	185
2.4. The groups $\Delta_\Phi^r(\Gamma, \mathfrak{A})$ and the dimension of Φ	186
2.5. Remark	186
3. The Homomorphism of $\Delta_\Phi^r(\Gamma, \mathfrak{A})$ into $\Delta_{\Phi'}^r(\Gamma', \mathfrak{A})$ Induced by a (Ψ, Ψ') -mapping $C_{\Phi'}^\Phi$	187
3.1. The homomorphism $C_{\Phi'}^\Phi$	187

Chapter

3.2. (Ψ, Ψ') -homologous and (Ψ, Ψ') -homotopic mappings; (Ψ, Ψ') -deformations	188
3.3. Deformation of a relative cycle of Φ	189
4. The Groups $\Delta_\Phi'(\Gamma)$ of Polyhedra Φ and Ψ	190
4.1. Introductory remarks	190
4.2. The fundamental theorem	190
4.3. The homomorphism $C_{\alpha'}^{\alpha}$ of $\Delta_{\Gamma_\alpha}' = \Delta'(K_\alpha \setminus K_{\Psi_\alpha}, \mathfrak{A})$ into $\Delta_{\Gamma'_\alpha}' = \Delta'(K_{\alpha'} \setminus K_{\Psi'_\alpha}, \mathfrak{A})$ induced by a (Ψ, Ψ') - mapping $C_{\Phi'}^\Phi$	192
4.4. Definition of the homology dimension of a polyhedron. Another proof of the invariance of the dimension number	192
4.5. The definition of the homology dimension of a compactum	193
5. Pseudomanifolds With Boundary	193
5.1. Orientation of a pseudomanifold with boundary	193
5.2. Introductory remarks; definition of the degree of a continuous mapping of a pseudomanifold with boundary	194
5.3. Some properties of the degree of a mapping	195
5.4. Examples	195
6. The Groups $\Delta_p'(\Phi)$ (The Local Δ' -groups of a Compactum Φ)	198
6.1. Definition of the groups $\Delta_p'(\Phi)$	198
6.2. The local character of the groups Δ_p'	199
7. The Local Δ -groups of Polyhedra	201
7.1. Notation and introductory remarks	201
7.2. The fundamental theorem	203
7.3. Application to the invariance of pseudomanifolds	209
APPENDIX 2	210
LIST OF SYMBOLS	238
INDEX	241