Contents

Dreface

Ticiacc	•
Introduction	1
CHAPTER 1	
Preliminaries	5
§1. The Problems of Representations and Their Solutions	5
§2. Methods	6
§3. The Contents of This Book	9
§4. References	11
§5. Problems	11
§6. Notation	11
CHAPTER 2	
Sums of Two Squares	13
§1. The One Square Problem	13
§2. The Two Squares Problem	13
§3. Some Early Work	14
§4. The Main Theorems	15
§5. Proof of Theorem 2	16
§6. Proof of Theorem 3	18
§7. The "Circle Problem"	20
§8. The Determination of $N_2(x)$	22
§9. Other Contributions to the Sum of Two Squares Problem	22
§10. Problems	23
CHAPTER 3	
Triangular Numbers and the Representation of Integers as Sums	
of Four Squares	24
§1. Sums of Three Squares	24
§2. Three Squares, Four Squares, and Triangular Numbers	25
§3. The Proof of Theorem 2	27
§4. Main Result	30
§5. Other Contributions	31
§6. Proof of Theorem 4	31
§7. Proof of Lemma 3	33

viii	Contents
§8. Sketch of Jacobi's Proof of Theorem 4	35
§9. Problems	36
CHAPTER 4	
Representations as Sums of Three Squares	38
§1. The First Theorem	38
§2. Proof of Theorem 1, Part I	38
§3. Early Results	39
§4. Quadratic Forms	39
§5. Some Needed Lemmas	42
§6. Proof of Theorem 1, Part II	46
§7. Examples	49
§8. Gauss's Theorem	51 53
§9. From Gauss to the Twentieth Century §10. The Main Theorem	54
§11. Some Results from Number Theory	55
§12. The Equivalence of Theorem 4 with Earlier Formulations	57
§13. A Sketch of the Proof of (4.7')	59
§14. Liouville's Method	60
§15. The Average Order of $r_3(n)$ and the Number of Representable	
Integers	61
§16. Problems	64
CHAPTER 5	
Legendre's Theorem	66
§1. The Main Theorem and Early Results	66
§2. Some Remarks and a Proof That the Conditions Are Necessary	67
§3. The Hasse Principle	68
§4. Proof of Sufficiency of the Conditions of Theorem 1	68
§5. Problems	71
CHAPTER 6	
Representations of Integers as Sums of Nonvanishing Squares	72
§1. Representations by $k \ge 4$ Squares	72
§2. Representations by k Nonvanishing Squares	72
§3. Representations as Sums of Four Nonvanishing Squares	74
§4. Representations as Sums of Two Nonvanishing Squares	75 75
§5. Representations as Sums of Three Nonvanishing Squares	73
§6. On the Number of Integers $n \le x$ That Are Sums of k Nonvanishing	79
Squares §7. Problems	83
CHAPTER 7	
CHAPTER 7	
The Problem of the Uniqueness of Essentially Distinct	0.4
Representations	84
§1. The Problem	84
§2. Some Preliminary Remarks	85

Contents	
§3. The Case $k = 4$	85
§4. The Case $k \ge 5$	86
§5. The Cases $k = 1$ and $k = 2$	87
§6. The Case $k=3$	88
§7. Problems	89
CHAPTER 8	
Theta Functions	91
§1. Introduction	91
§2. Preliminaries	91
§3. Poisson Summation and Lipschitz's Formula	92
§4. The Theta Functions	95
§5. The Zeros of the Theta Functions	97
§6. Product Formulae	99
§7. Some Elliptic Functions	101
§8. Addition Formulae	104
§9. Problems	105
CHAPTER 9	
Representations of Integers as Sums of an Even Number of	
Squares	107
§1. A Sketch of the Method	107
§2. Lambert Series	108
§3. The Computation of the Powers θ_3^{2k}	112
§4. Representation of Powers of θ_3 by Lambert Series	114
§5. Expansions of Lambert Series into Divisor Functions	117
§6. The Values of the $r_k(n)$ for Even $k \le 12$	121
§7. The Size of $r_k(n)$ for Even $k \le 8$	121
§8. An Auxilliary Lemma	123
§9. Estimate of $r_{10}(n)$ and $r_{12}(n)$	124
§10. An Alternative Approach	126
§11. Problems	127
CHAPTER 10	
Various Results on Representations as Sums of Squares	128
§1. Some Special, Older Results	128
§2. More Recent Contributions	129
§3. The Multiplicativity Problem	131
§4. Problems	133
CHAPTER 11	
Preliminaries to the Circle Method and the Method of Modular	
Functions	134
§1. Introduction	134
§2. Farey Series	136
§3. Gaussian Sums	137
§4. The Modular Group and Its Subgroups	139

· •	Contents

§5. Modular Forms §6. Some Theorems	143 145
§7. The Theta Functions as Modular Forms	146
§8. Problems	147
3-	
CHAPTER 12 The Circle Method	140
The Circle Method	149
§1. The Principle of the Method	149
§2. The Evaluation of the Error Terms and Formula for $r_s(n)$	153
§3. Evaluation of the Singular Series	156
$\S4$. Explicit Evaluation of $\mathscr S$	160
§5. Discussion of the Density of Representations	170
§6. Other Approaches	173
§7. Problems	173
CHAPTER 13	
Alternative Methods for Evaluating $r_s(n)$	175
§1. Estermann's Proof	175
§2. Sketch of the Proof by Modular Functions	178
§3. The Function $\psi_s(\tau)$	180
§4. The Expansion of $\psi_s(\tau)$ at the Cusp $\tau = -1$	182
§5. The Function $\theta^{s}(\tau)$	184
§6. Proof of Theorem 4	184
§7. Modular Functions and the Number of Representations by Quadratic	101
Forms	185
§8. Problems	186
30. 1.00.000	100
CHAPTER 14	
Recent Work	188
§1. Introduction	188
§2. Notation and Definitions	189
U	109
§3. The Representation of Totally Positive Algebraic Integers as Sums	192
of Squares	
§4. Some Special Results	194
§5. The Circle Problem in Algebraic Number Fields	196
§6. Hilbert's 17th Problem	197
§7. The Work of Artin	198
§8. From Artin to Pfister	201
§9. The Work of Pfister and Related Work	203
§10. Some Comments and Additions	207
§11. Hilbert's 11th Problem	208
§12. The Classification Problem and Related Topics	209
§13. Quadratic Forms Over \mathbb{Q}_p	212
§14. The Hasse Principle	216

Contents	xi
APPENDIX Open Problems	219
References	221
Bibliography	231
Addenda	242
Author Index	243
Subject Index	247