

TABLE OF CONTENTS

Editor's Preface	xiii
Author's Preface	xv
PART I: POINT TRANSFORMATIONS	
INTRODUCTORY CHAPTER: GROUP AND DIFFERENTIAL EQUATIONS	1
§ 1. Continuous groups	1
1.1 Topological groups	1
1.2 Lie groups	2
1.3 Local groups	4
1.4 Local Lie groups	5
§ 2. Lie algebras	6
2.1 Definitions	6
2.2 Lie algebras and local Lie groups	10
2.3 Inner automorphisms	12
2.4 The Levi-Mal'cev theorem	14
§ 3. Transformation groups	16
3.1 Local transformation groups	16
3.2 Lie's equation	17
3.3 Invariants	19
3.4 Invariant manifolds	20
§ 4. Invariant differential equations	23
4.1 Prolongation of point transformations	23
4.2 The defining equation	27
4.3 Invariant and partially invariant solutions	29
4.4 The method of invariant majorants	32

§ 5. Examples	40
5.1 Let $x \in \mathbb{R}^n$, $a \in \mathbb{R}$	40
5.2 Let us illustrate the algorithm for computing the group admitted by a differential equation by means of the example of a second-order equation	42
5.3 The Korteweg-de Vries equation	45
5.4 Consider the equation of motion of a polytropic gas	48
CHAPTER 1: MOTIONS IN RIEMANNIAN SPACES	53
§ 6. The general group of motions	53
6.1 Local Riemannian manifolds	53
6.2 Arbitrary motions in V_n	58
6.3 The defect of a group of motions in V_n	62
6.4 Invariant family of spaces	64
§ 7. Examples of motions	67
7.1 Isometries	67
7.2 Conformal motions	68
7.3 Motions with $\delta = 2$	72
7.4 Nonconformal motions with $\delta = 1$	75
7.5 Motions with given invariants	78
§ 8. Riemannian spaces with nontrivial conformal group	81
8.1 Conformally related spaces	81
8.2 Spaces of constant curvature	85
8.3 Conformally-flat spaces	88
8.4 Spaces with definite metric	90
8.5 Lorentzian spaces	92
§ 9. Group analysis of Einstein's equations	97
9.1 Harmonic coordinates	97
9.2 The group admitted by Einstein's equations	101
9.3 The Lie-Vessiot decomposition	104
9.4 Exact solutions	105

§ 10. Conformally-invariant equations of second order	113
10.1 Preliminaries	113
10.2 Linear equations in S_n	117
10.3 Semilinear equations in S_n	121
10.4 Equations admitting an isometry group of maximal order	127
10.5 The wave equation in Lorentzian spaces	128
CHAPTER 2: A GROUP-THEORETICAL APPROACH TO THE HUYGENS PRINCIPLE	133
§ 11. General considerations and some history of the problem	133
11.1 Hadamard's problem	133
11.2 Hadamard's criterion	135
11.3 The Mathisson-Asgeirsson Theorem	136
11.4 The necessary conditions of Günther and McLenaghan	139
11.5 The Lagnese-Stellmacher transformation	143
11.6 The present state of the art and generalizations of Hadamard's problem	148
§ 12. The wave equation in V_4	150
12.1 Computation of the geodesic distance in a plane-wave metric	150
12.2 Conformal invariance and the Huygens principle	157
12.3 The solution of the Cauchy problem	163
12.4 The case of a trivial conformal group	171
§ 13. The Huygens principle in V_{n+1}	173
13.1 Preliminary analysis of the solution	173
13.2 The Fourier transform of the Bessel function $J_0(a \mu)$	178

13.3	The descent method. Representation of solution for arbitrary n	181
13.4	Summary of the Huygens principle	185
13.5	Failure of the connection between Huygens' principle and conformal invariance	188

PART II: TANGENT TRANSFORMATIONS

CHAPTER 3:	INTRODUCTION TO THE THEORY OF LIE-BÄCKLUND GROUPS	190
§ 14.	Heuristic considerations	190
14.1	Contact transformations	190
14.2	Finite-order tangent transformations	195
14.3	Bianchi-Lie transformation	202
14.4	Bäcklund transformations. Examples	205
14.5	The concept of infinite-order tangent transformation	211
§ 15.	Formal groups	213
15.1	Lie's equation for formal one-parameter groups	213
15.2	Invariants and invariant manifolds	219
§ 16.	One-parameter groups of Lie-Bäcklund transformations	222
16.1	Definition and the infinitesimal criterion	222
16.2	Lie-Bäcklund operators. Canonical operators	229
16.3	Examples	232
§ 17.	Invariant differential manifolds	235
17.1	A criterion of invariance	235
17.2	Examples of solutions of the defining equation	239
17.3	Ordinary differential equations	242
17.4	The isomorphism theorem	246
17.5	Linearization by means of Lie-Bäcklund transformations	249

CHAPTER 4: EQUATIONS WITH INFINITE LIE-BÄCKLUND GROUPS	253
§ 18. Typical examples	253
18.1 The heat equation	253
18.2 The Korteweg-de Vries equation	258
18.3 A fifth-order equation	264
18.4 The wave equation	266
§ 19. Evolution equations	267
19.1 The algebra A_F	267
19.2 The Faà de Bruno formula	273
19.3 The algebra L_F	275
19.4 Differential substitutions	282
19.5 Equivalence transformations defined by ordinary differential equations	286
§ 20. Analysis of second- and third-order evolution equations	288
20.1 $m = 2$	288
20.2 $m = 3$	295
20.3 Two systems of nonlinear equations	301
§ 21. The equation $F(x,y,z,p,q,r,s,t) = 0$	303
21.1 Analysis of the general case	303
21.2 Classification of the equations $s = F(z)$	307
21.3 A system of two nonlinear equations	311
CHAPTER 5: CONSERVATION LAWS	314
§ 22. Fundamental theorems	314
22.1 The Noether identity	314
22.2 The Noether theorem	315
22.3 Invariance on the extremals	317
22.4 The action of the adjoint algebra	319
22.5 First integrals of evolution equations	321

§ 23. Examples	323
23.1 Motion in de Sitter space	323
23.2 The equation $u_{tt} + \Delta^2 u = 0$	326
23.3 The non-steady-state transonic gas flow	327
23.4 Short waves	331
§ 24. The Lorentz group	333
24.1 Conservation laws in relativistic mechanics	333
24.2 A nonlinear wave equation	336
24.3 Dirac equation	338
§ 25. The Galilean group	343
25.1 Motion of a particle	343
25.2 Perfect gas	347
25.3 Incompressible fluid	356
25.4 Shallow-water flow	360
25.5 A basis of conservation laws for the K-dV equation	361
REFERENCES	364
INDEX	385