Contents

Preface v

CHAPTER 1. **Basic Notions of Topology and the Value of Topological Reasoning**

1.1. Introduction 1
1.2. Basic topological notions 8
1.3. Homeomorphisms, homotopy and the idea of topological invariants 20
1.4. Topological invariants of compactness and connectedness . 22
1.5. Invariance of the dimension of $\mathbf{R}^n$ 23

CHAPTER 2. **Differential Geometry: Manifolds and Differential Forms**

2.1. Manifolds 25
2.2. Orientability 33
2.3. Calculus on manifolds 37
2.4. Infinite dimensional manifolds 49
2.5. Differentiable structures 49

CHAPTER 3. **The Fundamental Group**

3.1. Introduction 51
3.2. Definition of the fundamental group 56
3.3. Simplexes and the calculating theorem 67
3.4. Triangulation of a space with examples 70
3.5. Fundamental group of a product $X \times Y$ 77

CHAPTER 4. **The Homology Groups**

4.1. Introduction 79
4.2. Oriented simplexes and the definition of the homology groups . 83
4.3. Abelian groups 90
4.4. Relative homology groups 95
4.5. Exact sequences 99

4.6. Torsion, Kunneth formula, Euler–Poincaré formula and singular homology 104

CHAPTER 5. **The Higher Homotopy Groups**

5.1. Introduction 109
5.2. Definition of higher homotopy groups 109
5.3. Abelian nature of higher homotopy groups 112
5.4. Relative homotopy groups 113
5.5. The exact homotopy sequence 116

CHAPTER 6. **Cohomology and De Rham Cohomology**

6.1. Introduction 120
6.2. $H^p(M; \mathbf{R})$ and Poincaré's lemma 123
6.3. Poincaré's lemma 125
6.4. Calculation of $H^p(M; \mathbf{R})$ 127
6.5. General remarks 136
6.6. The cup product 137
6.7. Superiority of cohomology over homology 138

CHAPTER 7. **Fibre Bundles and Further Differential Geometry**

7.1. Introduction 140
7.2. Fibre bundle 141
7.3. More examples of bundles 148
7.4. When is a bundle trivial? 152
7.5. Sections of bundles and singularities of vector fields . . . 156
7.6. Cutting a bundle down to size: reduction of the group and contraction of the base space 159
7.7. Remarks on almost Hamiltonian and almost complex structures 165
7.8. G-structures on a compact closed manifold M 171
7.9. Lie derivative 171
7.10. Connection and curvature 174
7.11. The connection form and the gauge potential 177
7.12. Parallel transport, covariant derivative and curvature . . . 178
7.13. Covariant exterior derivatives 181
7.14. The Bianchi identities and *F 182
7.15. Connection in the tangent bundle 184
7.16. The torsion tensor 187
7.17. Geodesics 190

7.18. The Levi–Civita connection 191
7.19. The Yang–Mills connection 194
7.20. The Maxwell connection 196
7.21. General remarks 198
7.22. Characteristic classes 200
7.23. Chern, Pontrjagin and Euler classes 204
7.24. Characteristic classes in terms of curvature and invariant polynomials 206
7.25. Classification of bundles 211
7.26. The Stiefel–Whitney class 212
7.27. Calculation of characteristic classes 213
7.28. General remarks 217
7.29. Formulae obeyed by characteristic classes 219
7.30. Global invariants and local geometry 221

CHAPTER 8. **Morse Theory**

8.1. More inequalities 227
8.2. Morse lemma 229
8.3. Symmetry breaking selection rules in crystals 236
8.4. Estimating equilibrium positions 242

CHAPTER 9. **Defects, Textures and Homotopy Theory**

9.1. Planar spin in two dimensions 244
9.2. Definition of an ordered medium 245
9.3. Stability of defects theorem 246
9.4. Examples 250
9.5. General remarks and crossing of defects, textures and $\pi_3(S^2)$ 251

CHAPTER 10. **Yang–Mills Theories: Instantons and Monopoles**

10.1. Introduction 256
10.2. Instantons 259
10.3. Topology and boundary conditions 260
10.4. Instantons and absolute minima 263
10.5. The instanton solution 265
10.6. The instanton number and the second Chern class . . . 269
10.7. Multi-instantons 272
10.8. Quaternions and $SU(2)$ connections 272
10.9. The $k = 1$ instanton in terms of quaternions 276
10.10. Instantons with $|k| > 1$ and quaternions 278

10.11. Example of instantons with $|k|>1$ 281
10.12. Twistor methods and instantons 283
10.13. The projective twistor space 283
10.14. Twistor space and planes in $\mathbf{C}^4$ 285
10.15. α-planes and anti-self-dual connections 288
10.16. The equivalence between instantons and holomorphic vector bundles 289
10.17. Construction of an instanton given a holomorphic vector bundle 295
10.18. The Minkowski case 297
10.19. Monopoles 297
10.20. The Bohm–Aharanov effect 301

Further Reading 305

Subject Index 306