

Table of Contents

List of Frequently Used Symbols	XVII
Chapter 1. The Factorization of Integers	1
1.1 Divisibility	1
1.2 Prime Numbers and Composite Numbers	2
1.3 Prime Numbers	3
1.4 Integral Modulus	4
1.5 The Fundamental Theorem of Arithmetic	6
1.6 The Greatest Common Factor and the Least Common Multiple	7
1.7 The Inclusion-Exclusion Principle	10
1.8 Linear Indeterminate Equations	11
1.9 Perfect Numbers	13
1.10 Mersenne Numbers and Fermat Numbers	14
1.11 The Prime Power in a Factorial	16
1.12 Integral Valued Polynomials	17
1.13 The Factorization of Polynomials	19
Notes	21
Chapter 2. Congruences	22
2.1 Definition	22
2.2 Fundamental Properties of Congruences	22
2.3 Reduced Residue System	23
2.4 The Divisibility of $2^{p-1} - 1$ by p^2	24
2.5 The Function $\phi(m)$	26
2.6 Congruences	28
2.7 The Chinese Remainder Theorem	29
2.8 Higher Degree Congruences	31
2.9 Higher Degree Congruences to a Prime Power Modulus	32
2.10 Wolstenholme's Theorem	33
Chapter 3. Quadratic Residues	35
3.1 Definitions and Euler's Criterion	35
3.2 The Evaluation of Legendre's Symbol	36
3.3 The Law of Quadratic Reciprocity	38
3.4 Practical Methods for the Solutions	42

3.5 The Number of Roots of a Quadratic Congruence	44
3.6 Jacobi's Symbol	44
3.7 Two Terms Congruences	47
3.8 Primitive Roots and Indices	48
3.9 The Structure of a Reduced Residue System	49
Chapter 4. Properties of Polynomials	57
4.1 The Division of Polynomials	57
4.2 The Unique Factorization Theorem	58
4.3 Congruences	60
4.4 Integer Coefficients Polynomials	61
4.5 Polynomial Congruences with a Prime Modulus	62
4.6 On Several Theorems Concerning Factorizations	63
4.7 Double Moduli Congruences	64
4.8 Generalization of Fermat's Theorem	65
4.9 Irreducible Polynomials mod p	66
4.10 Primitive Roots	67
4.11 Summary	68
Chapter 5. The Distribution of Prime Numbers	70
5.1 Order of Infinity	70
5.2 The Logarithm Function	71
5.3 Introduction	72
5.4 The Number of Primes is Infinite	75
5.5 Almost All Integers are Composite	78
5.6 Chebyshev's Theorem	79
5.7 Bertrand's Postulate	82
5.8 Estimation of a Sum by an Integral	85
5.9 Consequences of Chebyshev's Theorem	89
5.10 The Number of Prime Factors of n	94
5.11 A Prime Representing Function	96
5.12 On Primes in an Arithmetic Progression	97
Notes	99
Chapter 6. Arithmetic Functions	102
6.1 Examples of Arithmetic Functions	102
6.2 Properties of Multiplicative Functions	104
6.3 The Möbius Inversion Formula	105
6.4 The Möbius Transformation	107
6.5 The Divisor Function	111
6.6 Two Theorems Related to Asymptotic Densities	113
6.7 The Representation of Integers as a Sum of Two Squares	115
6.8 The Methods of Partial Summation and Integration	120
6.9 The Circle Problem	122
6.10 Farey Sequence and Its Applications	125

6.11 Vinogradov's Method of Estimating Sums of Fractional Parts	129
6.12 Application of Vinogradov's Theorem to Lattice Point Problems	134
6.13 Ω -results	138
6.14 Dirichlet Series.	143
6.15 Lambert Series	146
Notes	147
Chapter 7. Trigonometric Sums and Characters	149
7.1 Representation of Residue Classes	149
7.2 Character Functions	151
7.3 Types of Characters	156
7.4 Character Sums	159
7.5 Gauss Sums	162
7.6 Character Sums and Trigonometric Sums	169
7.7 From Complete Sums to Incomplete Sums	170
7.8 Applications of the Character Sum $\sum_{x=1}^p \left(\frac{x^2 + ax + b}{p} \right)$	174
7.9 The Problem of the Distribution of Primitive Roots	177
7.10 Trigonometric Sums Involving Polynomials	180
Notes	185
Chapter 8. On Several Arithmetic Problems Associated with the Elliptic Modular Function	186
8.1 Introduction	186
8.2 The Partition of Integers	187
8.3 Jacobi's Identity	188
8.4 Methods of Representing Partitions	193
8.5 Graphical Method for Partitions	195
8.6 Estimates for $p(n)$	199
8.7 The Problem of Sums of Squares	204
8.8 Density	210
8.9 A Summary of the Problem of Sums of Squares	215
Chapter 9. The Prime Number Theorem	217
9.1 Introduction	217
9.2 The Riemann ζ -Function	219
9.3 Several Lemmas	222
9.4 A Tauberian Theorem	226
9.5 The Prime Number Theorem	231
9.6 Selberg's Asymptotic Formula	233
9.7 Elementary Proof of the Prime Number Theorem	235
9.8 Dirichlet's Theorem.	243
Notes	248

Chapter 10. Continued Fractions and Approximation Methods	250
10.1 Simple Continued Fractions	250
10.2 The Uniqueness of a Continued Fraction Expansion	252
10.3 The Best Approximation.	254
10.4 Hurwitz's Theorem.	255
10.5 The Equivalence of Real Numbers	257
10.6 Periodic Continued Fractions	260
10.7 Legendre's Criterion	261
10.8 Quadratic Indeterminate Equations	262
10.9 Pell's Equation	264
10.10 Chebyshev's Theorem and Khintchin's Theorem	266
10.11 Uniform Distributions and the Uniform Distribution of $n\vartheta \pmod{1}$	269
10.12 Criteria for Uniform Distributions	270
 Chapter 11. Indeterminate Equations	 276
11.1 Introduction	276
11.2 Linear Indeterminate Equations.	276
11.3 Quadratic Indeterminate Equations.	278
11.4 The Solution to $ax^2 + bxy + cy^2 = k$	278
11.5 Method of Solution	283
11.6 Generalization of Soon Go's Theorem	286
11.7 Fermat's Conjecture	288
11.8 Markoff's Equation	288
11.9 The Equation $x^3 + y^3 + z^3 + w^3 = 0$	290
11.10 Rational Points on a Cubic Surface	293
Notes	299
 Chapter 12. Binary Quadratic Forms	 300
12.1 The Partitioning of Binary Quadratic Forms into Classes	300
12.2 The Finiteness of the Number of Classes.	302
12.3 Kronecker's Symbol	304
12.4 The Number of Representations of an Integer by a Form	307
12.5 The Equivalence of Forms mod q	309
12.6 The Character System for a Quadratic Form and the Genus.	314
12.7 The Convergence of the Series $K(d)$	317
12.8 The Number of Lattice Points Inside a Hyperbola and an Ellipse .	318
12.9 The Limiting Average	318
12.10 The Class Number: An Analytic Expression.	321
12.11 The Fundamental Discriminants	322
12.12 The Class Number Formula.	323
12.13 The Least Solution to Pell's Equation	326
12.14 Several Lemmas	329
12.15 Siegel's Theorem	331
Notes	337

Chapter 13. Unimodular Transformations	338
13.1 The Complex Plane	338
13.2 Properties of the Bilinear Transformation	339
13.3 Geometric Properties of the Bilinear Transformation	342
13.4 Real Transformations	344
13.5 Unimodular Transformations	348
13.6 The Fundamental Region	350
13.7 The Net of the Fundamental Region	354
13.8 The Structure of the Modular Group.	355
13.9 Positive Definite Quadratic Forms	356
13.10 Indefinite Quadratic Forms	358
13.11 The Least Value of an Indefinite Quadratic Form.	361
Chapter 14. Integer Matrices and Their Applications	365
14.1 Introduction	365
14.2 The Product of Matrices	371
14.3 The Number of Generators for Modular Matrices	377
14.4 Left Association	382
14.5 Invariant Factors and Elementary Divisors	384
14.6 Applications	387
14.7 Matrix Factorizations and Standard Prime Matrices	389
14.8 The Greatest Common Factor and the Least Common Multiple	394
14.9 Linear Modules	399
Chapter 15. p-adic Numbers.	405
15.1 Introduction	405
15.2 The Definition of a Valuation	408
15.3 The Partitioning of Valuations into Classes	410
15.4 Archimedean Valuations.	411
15.5 Non-Archimedean Valuations.	412
15.6 The φ -Extension of the Rationals	415
15.7 The Completeness of the Extension	417
15.8 The Representation of p -adic Numbers	417
15.9 Application.	421
Chapter 16. Introduction to Algebraic Number Theory	423
16.1 Algebraic Numbers	423
16.2 Algebraic Number Fields	424
16.3 Basis.	425
16.4 Integral Basis	427
16.5 Divisibility	430
16.6 Ideals	431
16.7 Unique Factorization Theorem for Ideals	433
16.8 Basis for Ideals	436
16.9 Congruent Relations	437

16.10 Prime Ideals	438
16.11 Units	441
16.12 Ideal Classes	441
16.13 Quadratic Fields and Quadratic Forms.	442
16.14 Genus	445
16.15 Euclidean Fields and Simple Fields	447
16.16 Lucas's Criterion for the Determination of Mersenne Primes	449
16.17 Indeterminate Equations.	450
16.18 Tables	454
Notes	473
 Chapter 17. Algebraic Numbers and Transcendental Numbers	474
17.1 The Existence of Transcendental Numbers	474
17.2 Liouville's Theorem and Examples of Transcendental Numbers.	476
17.3 Roth's Theorem on Rational Approximations to Algebraic Numbers	478
17.4 Application of Roth's Theorem	478
17.5 Application of Thue's Theorem	480
17.6 The Transcendence of e	483
17.7 The Transcendence of π	486
17.8 Hilbert's Seventh Problem	488
17.9 Gelfond's Proof	490
Notes	493
 Chapter 18. Waring's Problem and the Problem of Prouhet and Tarry	494
18.1 Introduction	494
18.2 Lower Bounds for $g(k)$ and $G(k)$	494
18.3 Cauchy's Theorem.	496
18.4 Elementary Methods	499
18.5 The Easier Problem of Positive and Negative Signs	503
18.6 Equal Power Sums Problem	505
18.7 The Problem of Prouhet and Tarry	507
18.8 Continuation	511
 Chapter 19. Schnirelmann Density	514
19.1 The Definition of Density and its History	514
19.2 The Sum of Sets and its Density	515
19.3 The Goldbach-Schnirelmann Theorem.	518
19.4 Selberg's Inequality	519
19.5 The Proof of the Goldbach-Schnirelmann Theorem	525
19.6 The Waring-Hilbert Theorem.	528
19.7 The Proof of the Waring-Hilbert Theorem	530
Notes	534

Chapter 20. The Geometry of Numbers	535
20.1 The Two Dimensional Situation	535
20.2 The Fundamental Theorem of Minkowski	538
20.3 Linear Forms	540
20.4 Positive Definite Quadratic Forms	542
20.5 Products of Linear Forms	543
20.6 Method of Simultaneous Approximations	546
20.7 Minkowski's Inequality	547
20.8 The Average Value of the Product of Linear Forms	554
20.9 Tchebotaref's Theorem	556
20.10 Applications to Algebraic Number Theory	558
20.11 The Least Value for $ d $	561
Bibliography	565
Index	569