

Contents

Chapter 1 Introductory Topics – Scalars and Vectors

1.1	Determinants	19
1.1.1	Introduction	19
1.1.2	Second-Order Determinants	19
1.1.3	Third-Order Determinants	20
1.1.4	Fourth and Higher Order Determinants	21
1.1.5	Properties of Third-Order Determinants	22
1.1.6	Minors and Cofactors	23
1.1.7	Further properties of Third-Order Determinants	23
	Examples 1.1	25
	Problems 1.1	25
1.2	Matrices	26
1.2.1	Introduction	26
1.2.2	Matrix addition	27
1.2.3	Multiplication by a Scalar	27
1.2.4	Matrix Multiplication	27
1.2.5	Properties of Matrix Multiplication	29
1.2.6	Zero and Unit Matrices	29
1.2.7	The Transpose Matrix. Symmetric and Skew-Symmetric Matrices	30
1.2.8	The Inverse Matrix	31
1.2.9	The Rank of a Matrix	31
	Examples 1.2	32
	Problems 1.2	33
1.3	Scalars	34
1.3.1	Introduction	34
1.3.2	Scalar Fields	34
	Example 1.3	35
	Problems 1.3	35

Contents

1.4	Vectors	35
1.4.1	Introduction	35
1.4.2	Definition of a Real Vector Space	36
Examples 1.4	36	
Problems 1.4	37	
1.5	Vector Properties	37
1.5.1	Properties of Vectors in \mathbb{R}^3	37
1.5.2	The Scalar Product of Two Vectors	37
1.5.3	The Vector Product of Two Vectors	38
Example 1.5	39	
Problems 1.5	39	
1.6	The Geometrical Representation of Vectors	40
1.6.1	Directed Line Segments	40
1.6.2	The Parallelogram Law of Addition	40
1.6.3	The Triangle Law of Addition	41
1.6.4	The Geometrical Interpretation of the Scalar Product	42
1.6.5	Direction Cosines	44
1.6.6	The Geometrical Interpretation of the Vector Product	45
Examples 1.6	48	
Problems 1.6	49	
1.7	Products of More than Two Vectors	50
1.7.1	The Triple Scalar Product	50
1.7.2	The Triple Vector Product	50
1.7.3	Higher Products of Vectors	51
1.7.4	Linear Dependence and Independence	51
Examples 1.7	53	
Problems 1.7	53	
1.8	Vector Fields	54
1.8.1	Definition	54
1.8.2	Differentiation of Scalar Fields	54
Examples 1.8	55	
Problems 1.8	56	
1.9	The Summation Convention	56
1.9.1	Suffices	56
1.9.2	The Kronecker Delta	57
Examples 1.9	57	
Problems 1.9	58	
1.10	Transformations of Vectors	58
1.10.1	The Invariance of Vectors - Transformation Properties	58
1.10.2	Rotations	59
1.10.3	The Rotation Matrix	60
Examples 1.10	61	
Problems 1.10	62	

1.11 Rate of Change of a Vector in a Rotating Frame of Reference	63
Chapter 2 Second-Order Cartesian Tensors	
2.1 Introduction	66
2.1.1 Linear Vector Functions	66
2.1.2 Dyads	66
Examples 2.1	67
Problems 2.1	67
2.2 Properties of Second-order Tensors	68
2.2.1 Properties of Dyads	68
2.2.2 The Components of a Dyad	69
2.2.3 The Components of the Dot Product of a Tensor and a Vector	69
2.2.4 Properties of Second-Order Tensors	70
Examples 2.2	70
Problems 2.2	71
2.3 Further Properties of Second-Order Tensors	72
2.3.1 Pre- and Post-Multiplication of a Tensor and a Vector	72
2.3.2 The Transpose Tensor	72
Examples 2.3	73
Problems 2.3	73
2.4 Symmetric and Antisymmetric Tensors	74
Examples 2.4	76
Problems 2.4	76
2.5 Transformation Properties of Tensors	76
2.5.1 The Invariance of Tensors - Transformation Law	76
2.5.2 The Practical Transformation of Tensor Components	78
Example 2.5	78
Problems 2.5	79
2.6 The Scalar Invariants of a Second-Order Tensor	80
Examples 2.6	82
Problems 2.6	82
2.7 Inner Products	82
2.7.1 The Inner Product of Two Tensors - Contraction	82
2.7.2 The Double Inner Product of Two Tensors	83
Example 2.7	84
Problems 2.7	84
2.8 Vectors and Tensors with Complex Components	84
Example 2.8	85
Problems 2.8	85
2.9 The Eigenvector Problem	86
2.9.1 Statement of the Problem	86

Contents

2.9.2	Eigenvectors of the Hermitian Transpose Tensor	88
	Example 2.9	89
	Problems 2.9	90
2.10	The Eigenvector Problem: Degenerate Cases	91
2.10.1	Double Root of the Characteristic Equation	91
2.10.2	Treble Root of the Characteristic Equation	94
	Examples 2.10	95
	Problem 2.10	97
2.11	Hermitian Tensors	97
2.11.1	The Eigenvectors of Hermitian Tensors	97
2.11.2	Hermitian Tensors: The Degenerate Cases	98
	Example 2.11	100
	Problem 2.11	100
2.12	Real Symmetric Tensors	101
2.12.1	The Eigenvectors of Real Symmetric Tensors	101
2.12.2	Diagonalization of a Symmetric Tensor	101
	Examples 2.12	102
	Problems 2.12	103
2.13	The Cayley - Hamilton Theorem	103
	Example 2.13	105
	Problems 2.13	105
2.14	The Representation Theorem for Symmetric Tensors	106
2.15	Tensor Fields	108
2.15.1	Definition	108
2.15.2	Differentiation of Vector Fields	109
	Examples 2.15	110
	Problems 2.15	111

Chapter 3 Third-Order Cartesian Tensors

3.1	Tensors of the Third-Order	112
3.1.1	Definition	112
3.1.2	Products Involving Third-Order Tensors	113
3.1.3	The Transformation Law for Third-Order Tensors	113
	Example 3.1	113
	Problem 3.1	114
3.2	The Alternate Tensor	114
3.2.1	Definition	114
3.2.2	Applications of the Alternate Tensor	115
3.2.3	Differential Functions involving the Alternate Tensor	116
	Examples 3.2	117
	Problems 3.2	118
3.3	Cross Products of Tensors and Vectors	119
3.3.1	Definition	119

3.3.2	Rate of Change of a Tensor in a Rotating Frame of Reference	121
	Examples 3.3	122
	Problems 3.3	122
3.4	The Geometry of Rotations	123
3.4.1	General Rotations	123
3.4.2	Rotations about the Coordinate Axes	126
	Examples 3.4	128
	Problems 3.4	129
3.5	Integral Theorems in Tensor Form	130
3.5.1	The Tensor Form of Green's Theorem	130
3.5.2	Special Cases of Green's Theorem	130
3.5.3	The Tensor Form of Stokes's Theorem	131
3.5.4	Special Cases of Stokes's Theorem	131
	Problems 3.5	132
3.6	Curvilinear Coordinates	132
3.6.1	Introduction	132
3.6.2	Cartesian Tensors Referred to Orthogonal Curvilinear Coordinates	134
3.6.3	Spherical Polar Coordinates	137
3.6.4	Cylindrical Polar Coordinates	140
	Examples 3.6	141
	Problems 3.6	142
3.7	Third-Order Tensor Fields	143
	Examples 3.7	144

Chapter 4 Fourth-Order Cartesian Tensors

4.1	Tensors of the Fourth-Order - Definition	145
4.2	Isotropic Tensors	145
4.2.1	Definition	145
4.2.2	The Most General Isotropic Tensor of the Second-Order	146
4.2.3	The Most General Isotropic Tensor of the Third-Order	146
4.2.4	The Most General Isotropic Tensor of the Fourth-Order	147
4.2.5	Complex Isotropic Vectors	149
	Examples 4.2	150
	Problems 4.2	151

Chapter 5 The Inertia Tensor

5.1	The Inertia Tensor	152
5.1.1	Definition	152
5.1.2	Use of the Inertia Tensor - Kinetic Energy	153
5.1.3	The Moment of Inertia	155
5.1.4	Radius of Gyration	155
5.1.5	Momental Ellipsoid	156

Examples 5.1	156
Problems 5.1	158
5.2 Properties of the Inertia Tensor	159
5.2.1 The Components of the Inertia Tensor	159
5.2.2 The Parallel Axis Theorem	159
5.2.3 The Perpendicular Axes Theorem for Laminae	162
Examples 5.2	163
Problems 5.2	163
5.3 Calculation of the Inertia Tensor of Rectangular Bodies	164
5.3.1 Introduction	164
5.3.2 The Inertia Tensor of a Uniform Rectangular Block	165
5.3.3 Equimomental Systems	167
5.3.4 The Inertia Tensor of a Uniform Rectangular Block – An Alternative Approach	168
Examples 5.3	171
Problems 5.3	172
5.4 The Inertia Tensor of Non-Rectangular Bodies	173
5.4.1 The Inertia Tensor of a Uniform Parallelogram Lamina	173
5.4.2 The Inertia Tensor of a Uniform Solid Parallelepiped	174
5.4.3 The Inertia Tensor of a Uniform Triangular Lamina	175
5.4.4 The Inertia Tensor of a Uniform Solid Tetrahedron	177
Example 5.4	179
Problems 5.4	180
5.5 The Inertia Tensor of Bodies with Curved Boundaries	180
5.5.1 The Inertia Tensor of a Ring	180
5.5.2 The Inertia Tensor of a Circular Lamina	181
5.5.3 The Inertia Tensor of a Hollow Right Circular Cylinder	182
5.5.4 The Inertia Tensor of a Uniform Solid Right Circular Cylinder	184
5.5.5 The Inertia Tensor of a Uniform Elliptic Ring	185
5.5.6 The Inertia Tensor of an Elliptic Lamina	189
5.5.7 The Inertia Tensor of a Hollow Torus	190
5.5.8 The Inertia Tensor of a Solid Torus	192
5.5.9 The Inertia Tensor of a Spherical Shell	193
5.5.10 The Inertia Tensor of a Solid Sphere	194
Examples 5.5	195
Problems 5.5	197
5.6 Euler's Equations of Motion	199
Examples 5.6	201
Problems 5.6	203

Chapter 6 The Application of Cartesian Tensors to Fluid Mechanics

6.1 The Stress Tensor	204
--	------------

6.1.1	Introduction	204
6.1.2	The Stress Tensor	205
6.1.3	Properties of the Stress Tensor	207
6.1.4	The Appearance of the Stress Tensor in the Equation of Fluid Motion	208
6.1.5	The Symmetry of the Stress Tensor	209
6.1.6	Decomposition of the Stress Tensor	210
	Examples 6.1	211
	Problems 6.1	217
6.2	Analysis of Fluid Motion	219
6.2.1	The Rate of Strain Tensor	219
6.2.2	The Contribution of the Rate of Strain Tensor to the Relative Velocity	220
6.2.3	The Rate of Strain	220
6.2.4	The Contribution of the Antisymmetric Tensor ξ to the Relative Velocity	221
	Examples 6.2	222
	Problems 6.2	226
6.3	The Navier - Stokes Equations of Fluid Motion	227
Chapter 7 The Application of Cartesian Tensors to Elasticity		
7.1	Strain	229
7.1.1	The Strain Tensor - Infinitesimal Strain	229
7.1.2	The Contribution of the Strain Tensor to the Deformation	231
7.1.3	The Contribution of the Antisymmetric Tensor ζ to the Deformation	232
7.1.4	The Compatibility Equations	232
	Examples 7.1	233
	Problems 7.1	235
7.2	The Relation between Stress and Strain for Elastic Solids	236
7.2.1	Hooke's Law	236
7.2.2	The Generalized Hooke's Law	238
7.2.3	Simple Shear	239
7.2.4	Isotropic Expansion	239
7.2.5	The Extension of a Wire	239
	Examples 7.2	241
	Problems 7.2	243
7.3	Strain Energy	244
7.3.1	The Strain Energy Function	244
7.3.2	Energy Stored in Dilatation and Distortion	245
	Example 7.3	246
	Problems 7.3	246
7.4	The Displacement Equation	247
	Examples 7.4	248

Problems 7.4	252
7.5 The Rotating Disc Problem	253
7.6 Finite Deformation	257
7.6.1 Lagragian Finite Strain Components	257
7.6.2 The Eulerian Finite Strain Components	258
7.6.3 The Relation between the Eulerian Strain Tensor and the Rate - of - Strain Tensor	259
Examples 7.6	261
Problems 7.6	263
Appendix	
A.1 General Tensors	266
A.1.1 Contravariant and Covariant Components	266
A.1.2 The Metric Tensor	268
A.1.3 The Quotient Theorem	269
A.2 Differentiation of General Tensors	270
A.2.1 The Christoffel Symbols	270
A.2.2 The Transformation Law for Christoffel Symbols	270
A.2.3 The Covariant Differentiation of Vectors	271
Examples A.1	272
Answers to Problems	276
Bibliography	293
Index	294