

# CONTENTS

## PART I

### METRIC SPACES

#### CHAPTER I. PRELIMINARY NOTIONS

1. Abstract sets . . . . .	1
2. Operations on sets. Lattices . . . . .	2
3. Distributivity. Boolean algebra . . . . .	4
4. Abstract spaces . . . . .	5
5. Semimetric spaces . . . . .	7
6. Topology of semimetric spaces . . . . .	8
7. An anomalous property of distance in semimetric spaces . . . . .	12
8. The triangle inequality. Metric spaces . . . . .	13
9. Examples of metric spaces . . . . .	15
10. Topology of metric spaces . . . . .	26

#### CHAPTER II. METRIC SEGMENTS AND LINES

11. Historical remarks . . . . .	32
12. Betweenness in metric spaces. Congruence . . . . .	33
13. Characterization of metric betweenness . . . . .	36
14. Convexity in metric spaces. Metric segments . . . . .	40
15. Two characteristic properties of metric segments . . . . .	44
16. Lemmas concerning closed and compact subsets. An equilateral triple theorem . . . . .	46
17. An equilateral triple characterization of metric segments . . . . .	48
18. Criteria for unique metric segments . . . . .	49
19. Convex extension of a set . . . . .	51
20. Passing points and terminal points . . . . .	53
21. Metric lines . . . . .	55

#### CHAPTER III. CURVE THEORY

22. Arcs and arc length . . . . .	59
23. Homogeneous $\epsilon$ -chains and $\delta$ -density . . . . .	60
24. First sharpening of length concept . . . . .	61
25. Second sharpening of length concept. Lower semi-continuity of arc length . . . . .	63
26. A relaxing of the length concept . . . . .	67
27. Continuous curves . . . . .	68
28. Existence of geodesic arcs . . . . .	70
29. The $n$ -lattice theorem . . . . .	73
30. Metric definitions of curvature . . . . .	74
31. Some properties of $K_M$ , $K_A$ , and $K_H$ . . . . .	77
32. Ptolemaic spaces. A lemma . . . . .	78
33. Segments characterized as arcs with vanishing Menger curvature . . . . .	80
34. Metrization of torsion . . . . .	84
35. Wald's metrization of Gauss curvature . . . . .	88

## PART II

## EUCLIDEAN AND HILBERT SPACES

## CHAPTER IV. CONGRUENT IMBEDDING IN EUCLIDEAN SPACE

36. Two fundamental problems of distance geometry . . . . .	90
37. Congruence indices . . . . .	91
38. Congruence order of the $E_n$ . . . . .	93
39. A set of sufficient conditions for congruence indices $(m+3, k)$ . . . . .	95
40. The Cayley-Menger determinant . . . . .	97
41. Imbedding a semimetric $(r+1)$ -tuple irreducibly in $E_r$ . . . . .	99
42. A solution of the euclidean imbedding problem . . . . .	101
43. Additional criteria for imbedding finite semimetric spaces in $E_r$ . . . . .	105
44. An example of a pseudo- $E_n$ space . . . . .	109
45. The structure of pseudo- $E_n$ $(n+3)$ -tuples . . . . .	110
46. Quasi-congruence order of $E_n$ . . . . .	115
47. Free $(n+2)$ -tuples . . . . .	118

## CHAPTER V. METRIC AND VECTOR CHARACTERIZATIONS OF EUCLIDEAN AND HILBERT SPACES. IMBEDDING IN HILBERT SPACE. NORMED LINEAR SPACES

48. First metric characterization of the $E_n$ . . . . .	122
49. The weak euclidean four-point property. Some lemmas . . . . .	123
50. Second metric characterization of the $E_n$ . . . . .	126
51. The pythagorean property . . . . .	129
52. Metric transforms and the euclidean four-point property . . . . .	130
53. Congruent imbedding in Hilbert space . . . . .	132
54. Metric transforms of euclidean and Hilbert spaces . . . . .	134
55. Metric characterizations of Hilbert space . . . . .	136
56. Normed linear spaces, inner product, and quasi inner product space . . . . .	137
57. Quasi inner product space satisfying Schwarz inequalities . . . . .	139
58. Generalized euclidean space in terms of a quasi inner product . . . . .	144

## CHAPTER VI. CONGRUENCE INDICES OF SOME EUCLIDEAN SUBSETS

59. Introductory remarks . . . . .	148
60. Hyperfinite and transfinite congruence orders of linear sets. A general theorem . . . . .	149
61. One-dimensional subsets of $E_2$ . . . . .	152
62. Two-dimensional subsets of $E_2$ . Monomorphic sets. Characterization of the circular disk . . . . .	156

## PART III

## THE NON-EUCLIDEAN SPACES

## CHAPTER VII. IMBEDDING AND CHARACTERIZATION THEOREMS FOR SPHERICAL SPACE. CHARACTERIZATION OF PSEUDO-SPHERICAL SETS

63. Solution of the imbedding problem for $S_{n,r}$ . . . . .	162
64. The $\Sigma_r$ space. Some lemmas . . . . .	163

65. Additional lemmas. Subspaces of $\Sigma_r$ . . . . .	167
66. Metric characterization of $S_{n,r}$ . . . . .	168
67. Basic properties of $S_{n,r}$ . . . . .	171
68. Derived properties of $S_{n,r}$ . . . . .	172
69. Pseudo- $S_{k,r}$ ( $k+3$ )-tuples, $k \leq n$ . . . . .	175
70. Some properties of pseudo- $S_{k,r}$ ( $k+4$ )-tuples without diametral point-pairs . . . . .	181
71. Characterization of pseudo- $S_{n,r}$ sets without diametral point-pairs . . . . .	186
72. Characterization of general pseudo- $S_{n,r}$ sets . . . . .	189

#### CHAPTER VIII. INTERSECTION THEOREMS FOR CONVEX SUBSETS OF $S_{n,r}$ . CONGRUENCE INDICES OF SPHERICAL CAPS

73. Preliminary definitions and remarks . . . . .	192
74. Some lemmas . . . . .	194
75. Intersection theorems . . . . .	198
76. Congruence indices of hemispheres and small caps . . . . .	202

#### CHAPTER IX. IMBEDDING AND CHARACTERIZATION THEOREMS FOR ELLIPTIC SPACE

77. The notion of $\delta$ -supplementation . . . . .	206
78. Imbedding theorems for $\mathcal{E}_{n,r}$ . . . . .	207
79. Some metric peculiarities of elliptic space . . . . .	209
80. Equilateral subsets . . . . .	211
81. The space problem for the $\mathcal{E}_{n,r}$ . A lemma . . . . .	214
82. Definition and elementary properties of an $\mathcal{E}_r$ space . . . . .	215
83. One-dimensional subspaces (lines) of $\mathcal{E}_r$ . . . . .	219
84. Imbedding line-sums of $\mathcal{E}_r$ in the elliptic plane . . . . .	221
85. Linear subspaces of $\mathcal{E}_r$ . . . . .	225
86. Characterization theorems . . . . .	227

#### CHAPTER X. CONGRUENCE AND SUPERPOSABILITY IN ELLIPTIC SPACE

87. Preliminary remarks and definitions . . . . .	231
88. Congruent subsets, one of which is contained in $\mathcal{E}_{1,r}$ . . . . .	232
89. Non-superposable congruent subsets of $\mathcal{E}_{n,r}$ . . . . .	233
90. Spherical sets and matrices associated with an elliptic $m$ -tuple . . . . .	235
91. First superposability theorems . . . . .	237
92. Congruent $m$ -tuples with corresponding triples superposable . . . . .	239
93. Superposability of infinite subsets of $\mathcal{E}_{n,r}$ . . . . .	241
94. First reduction theorem . . . . .	243
95. Two lemmas . . . . .	245
96. Second reduction theorem . . . . .	247
97. Superposability order. Pseudo $f$ -superposable sets . . . . .	251

## CHAPTER XI. METRIC-THEORETIC PROPERTIES OF THE ELLIPTIC PLANE. CONGRUENCE ORDER

98. Preliminary remarks . . . . .	255
99. Orthocentric quadruples . . . . .	255
100. Singular loci . . . . .	258
101. Freely movable quintuples . . . . .	259
102. A 'crowding' theorem for the elliptic plane . . . . .	262
103. Congruence order of $\mathcal{E}_{2,r}$ with respect to the class of semimetric spaces	264
104. Congruence invariance of metric bases. Congruence indices of the cross	270

## CHAPTER XII. GENERALIZED HYPERBOLIC SPACE

105. Introductory remarks . . . . .	273
106. The $(n+2)$ -point relation in $\mathcal{H}_{n,r}$ . . . . .	273
107. Generalized hyperbolic space. Three postulates . . . . .	275
108. Subspaces of $\mathcal{H}_{n,r}^\phi$ . . . . .	276
109. Some metric properties of $\mathcal{H}_{n,r}^\phi$ . . . . .	278
110. Additional metric properties of $\mathcal{H}_{k,r}^\phi$ . . . . .	282
111. Two existence postulates. Imbedding theorems for $\mathcal{H}_{n,r}^\phi$ . . . . .	284

## PART IV

### APPLICATIONS OF DISTANCE GEOMETRY

Foreword . . . . .	288
--------------------	-----

## CHAPTER XIII. METRIC METHODS IN DETERMINANT THEORY

112. Cayley-Menger determinants . . . . .	290
113. Determinants of type $\Delta$ . . . . .	294
114. Determinants of types $\Delta_{NP}$ , $\Delta_{NN}$ , and $\Lambda$ . . . . .	297
115. Quasi rank of $\Delta_{NN}$ determinants . . . . .	301

## CHAPTER XIV. METRIC METHODS IN LINEAR INEQUALITIES

116. Coefficient sets $C$ and solution sets $\Sigma(C)$ . . . . .	304
117. Some elementary properties of $C$ and $\Sigma(C)$ . . . . .	305
118. The generalized Minkowski theorem . . . . .	308
119. Coincidence of $C$ and $\Sigma(C)$ . . . . .	311
120. Existence theorems for solutions of a system $(I)$ . . . . .	312
121. Systems of strict inequalities . . . . .	313

## CHAPTER XV. METRIC METHODS IN LATTICE THEORY

122. Introduction . . . . .	315
123. Lemmas from lattice theory . . . . .	315
124. Lattice characterizations of metric betweenness . . . . .	317
125. Betweenness in normed distributive lattices. Betweenness in arbitrary lattices . . . . .	320
126. Lattice characterization of pseudo-linear quadruples . . . . .	323
127. Congruent imbedding of normed lattices in convex normed lattices . . . . .	325
128. Congruence of $D(L)$ with a euclidean subset . . . . .	327

129. Properties of the associated metric space of a normed lattice . . . . .	328
130. Characterization of $D(L)$ . . . . .	330
131. Autometrized Boolean algebras. Elementary properties of distance . . . . .	331
132. Betweenness in autometrized Boolean algebras . . . . .	333
133. The group of motions of $B$ . . . . .	334
134. Free mobility in an autometrized Boolean algebra . . . . .	336
135. Congruence order of $B$ with respect to the class of $B$ -metrized spaces . . . . .	337
BIBLIOGRAPHY . . . . .	339
INDEX . . . . .	344