# Contents


Preface v

Index of Symbols ix

**Chapter 1. MEASURES** 1

Section **0.** Set Theoretic Notations and Terminology 1

**1.** Rings and $\sigma$-Rings 3

**2.** The Lemma on Monotone Classes 6

**3.** Set Functions, Measures 8

**4.** Some Properties of Measures 11

**5.** Outer Measures 13

**6.** Extension of Measures 19

***7.** Lebesgue Measure 22

***8.** Measurable Covers 24

***9.** Completion of a Measure 28

**10.** The LUB of an Increasingly Directed Family of Measures 31

**Chapter 2. MEASURABLE FUNCTIONS** 35

**11.** Measurable Spaces 35

**12.** Measurable Functions 36

**13.** Combinations of Measurable Functions 40

**14.** Limits of Measurable Functions 43

**15.** Localization of Measurability 46

**16.** Simple Functions 48

**Chapter 3. SEQUENCES OF MEASURABLE FUNCTIONS** 52

Section **17.** Measure Spaces 52
**18.** The "Almost Everywhere" Concept 55
**19.** Almost Everywhere Convergence 57
**20.** Convergence in Measure 59
***21.** Almost Uniform Convergence, Egoroff's Theorem 64

**Chapter 4. INTEGRABLE FUNCTIONS** 70

**22.** Integrable Simple Functions 70
**23.** Heuristics 75
**24.** Nonnegative Integrable Functions 76
**25.** Integrable Functions 81
**26.** Indefinite Integrals 88
**27.** The Monotone Convergence Theorem 90
**28.** Mean Convergence 96

**Chapter 5. CONVERGENCE THEOREMS** 100

**29.** Dominated Convergence in Measure 100
**30.** Dominated Convergence Almost Everywhere 102
**31.** The $\mathscr{L}^1$ Completeness Theorem 103
**32.** Fatou's Lemma 105
**33.** The Space $\mathscr{L}^2$, Riesz-Fischer Theorem 107

**Chapter 6. PRODUCT MEASURES** 114

**34.** Rectangles 114
**35.** Cartesian Product of Two Measurable Spaces 117
**36.** Sections 119
**37.** Preliminaries 123
**38.** The Product of Two Finite Measure Spaces 124
**39.** The Product of Any Two Measure Spaces 127
**40.** Product of Two $\sigma$-Finite Measure Spaces; Iterated Integrals 134
**41.** Fubini's Theorem 142
***42.** Complements 144

**Chapter 7. FINITE SIGNED MEASURES** 149

Section **43.** Absolute Continuity 149

**44.** Finite Signed Measures 151

**45.** Contractions of a Finite Signed Measure 153

**46.** Purely Positive and Purely Negative Sets 154

**47.** Comparison of Finite Measures 156

**48.** A Preliminary Radon-Nikodym Theorem 159

**49.** Jordan-Hahn Decomposition of a Finite Signed Measure 162

**50.** Domination of Finite Signed Measures 165

**51.** The Radon-Nikodym Theorem for a Finite Measure Space 167

**52.** The Radon-Nikodym Theorem for a $\sigma$-Finite Measure Space 167

***53.** Riesz Representation Theorem 169

**Chapter 8. INTEGRATION OVER LOCALLY COMPACT SPACES** 173

**54.** Continuous Functions with Compact Support 173

**55.** $G_\delta$'s and $F_\sigma$'s 174

**56.** Baire Sets 176

**57.** Borel Sets 181

**58.** Preliminaries on Rings 183

**59.** Regularity 186

**60.** Regularity of Baire Measures 192

**61.** Regularity (*Continued*) 194

**62.** Regular Borel Measures 200

**63.** Contents 204

**64.** Regular Contents 210

**65.** The Regular Borel Extension of a Baire Measure 212

**66.** Integration of Continuous Functions with Compact Support 214

**67.** Approximation of Baire Functions 218

***68.** Approximation of Borel Functions 220

**69.** The Riesz-Markoff Representation Theorem 223

***70.** Completion Regularity 230

**Chapter 9. INTEGRATION OVER LOCALLY COMPACT GROUPS** 235

Section **71.** Topological Groups 235
**72.** Translates, Haar Integral 237
**73.** Translation Ratios 238
**74.** Existence of a Haar Integral 242
**75.** A Topological Lemma 248
**76.** Uniqueness of the Haar Integral 250
**77.** The Modular Function 256
**78.** Haar Measure 259
**79.** Translates of Integrable Functions 263
**80.** Adjoints of Continuous Functions with Compact Support 265
**81.** Convolution of Continuous Functions with Compact Support 266
**82.** Adjoints of Integrable Functions 269
**83.** The operation $f \triangledown g$ 273
**84.** Convolution of Integrable Baire Functions 275
**85.** Associativity of Convolution 280
***86.** The Group Algebra 284
***87.** Convolution of Integrable Simple Baire Functions 288
**88.** The domain of $f * g$ 292
***89.** Convolution of Integrable Borel Functions 295
***90.** Complements on Haar Measure 299
References and Notes 301
Bibliography 305
Index 308