

Contents

<i>Frontispiece</i>	<i>page</i>	ii
<i>Preface</i>		ix
1. Regular polygons		
1.1 Isometries		1
1.2 The cyclic and dihedral groups		1
1.3 The theorem of Leonardo da Vinci		3
1.4 The product of two involutory isometries		3
1.5 Regular polygons in n dimensions		3
1.6 Straight and circular polygons		4
1.7 Zigzags and antiprismatic polygons		6
1.8 Helical polygons		7
1.9 Remarks		8
2. Regular polyhedra		
2.1 Spherical tessellations		9
2.2 Flags and Petrie polygons		12
2.3 Reflection groups and rotation groups		14
2.4 Wythoff's construction		16
2.5 The Schwarz triangles		19
2.6 Remarks		20
3. Polyhedral kaleidoscopes		
3.1 The characteristic orthoscheme		21
3.2 The icosahedral kaleidoscope		23
3.3 Cayley diagrams and presentations		24
3.4 Finite groups generated by half-turns		26
3.5 Remarks		28
4. Real four-space and the unitary plane		
4.1 Spherical honeycombs		29
4.2 The crystallographic regular polytopes		30
4.3 Flags and orthoschemes		32
4.4 The spherical torus		35
4.5 Double prisms		37

4·6	The 600-cell and the 120-cell	38
4·7	The ten star polytopes	46
4·8	A family of regular complex polygons	46
4·9	Remarks	52
5. Frieze patterns		
5·1	Some examples	55
5·2	Proof of the periodicity	56
5·3	Ptolemaic patterns	57
5·4	Real polytopes in four dimensions	57
5·5	Different patterns for the same polytope	60
5·6	Patterns of order 6 and period 3	60
5·7	Real polytopes in n dimensions	62
5·8	Remarks	63
6. The geometry of quaternions		
6·1	Pairs of complex numbers	64
6·2	Quaternions of real numbers	65
6·3	Reflections	66
6·4	Rotations	66
6·5	Finite groups of quaternions	67
6·6	Generators for $\langle p, q, 2 \rangle$	68
6·7	Screws in Euclidean 4-space	69
6·8	Rotatory-reflections	72
6·9	Remarks	72
7. The binary polyhedral groups		
7·1	The cyclic and dicyclic groups	74
7·2	The binary tetrahedral group	75
7·3	The binary octahedral group	77
7·4	The binary icosahedral group	78
7·5	Finite groups generated by pure quaternions	78
7·6	Representation by matrices	80
7·7	The unimodular group	81
7·8	A representation using residues modulo $h + 1$	81
7·9	Remarks	82

Contents

8. Unitary space	
8·1 Affine coordinates	page 83
8·2 Hermitian forms	83
8·3 Inner products	84
8·4 Lengths and angles	84
8·5 Unitary transformations	85
8·6 Dual bases	86
8·7 Reflections	87
8·8 A complex kaleidoscope	88
8·9 The two-dimensional case	88
9. The unitary plane, using quaternions	
9·1 Unitary groups	89
9·2 A combination of cyclic groups	90
9·3 An extension of the binary polyhedral groups	90
9·4 Reflections	91
9·5 Groups generated by involutory reflections	91
9·6 Other groups generated by three reflections	93
9·7 Two-generator subgroups	93
9·8 The group $p_1[q] p_2$ and its invariant Hermitian form	94
9·9 Remarks	96
10. The complete enumeration of finite reflection groups in the unitary plane	
10·1 The finite unitary groups in the plane	98
10·2 Reflection groups of type 1	98
10·3 Reflection groups of types 2 and 3	99
10·4 Reflection groups of types 3' and 4	99
10·5 Reflection groups of type 5	100
10·6 Reflection groups of type 6	100
10·7 Reflection groups of type 7	101
10·8 Reflection groups of type 8	101
10·9 Reflection groups of type 9	101
11. Regular complex polygons and Cayley diagrams	
11·1 Regular complex polygons	103
11·2 Real representations	104
11·3 Petrie polygons	105
11·4 Some useful subgroups of $p[2q] r$	106
11·5 Cayley diagrams for reflection groups	108
11·6 Apeirogons	111

11·7	A general treatment for the binary polyhedral groups	112
11·8	Remarks	113
12. Regular complex polytopes defined and described		
12·1	Definitions	115
12·2	Hermitian forms	117
12·3	The Hessian polyhedron	119
12·4	Other complex polyhedra	124
12·5	The Witting polytope	132
12·6	The honeycomb of Witting polytopes	135
12·7	Cartesian products of apeirogons	135
12·8	Cycles of honeycombs	136
12·9	Remarks	140
13. The regular complex polygons and their symmetry groups		
13·1	The regular polytopes and their van Oss polygons	141
13·2	The regular honeycombs	144
13·3	Cycles and frieze patterns	146
13·4	Presenting the symmetry groups	147
13·5	A historical digression	149
13·6	Petrie polygons and exponents	150
13·7	Numerical properties of the non-starry polytopes	153
13·8	Presenting the collineation groups	154
13·9	Invariants	154
Tables		
TABLE I	The Schwarz triangles	156
TABLE II	The finite groups generated by two reflections	156
TABLE III	The two-dimensional reflection groups and their reflection subgroups	157
TABLE IV	The regular polygons	158
TABLE V	The non-starry polyhedra and four-dimensional polytopes	160
TABLE VI	The regular honeycombs	161
	<i>Answers to exercises</i>	162
	<i>Bibliography</i>	180
	<i>Index</i>	182