# Contents

**Chapter 1. THE ALGEBRA OF THE REAL NUMBERS**

1.1 Introduction . . . . . 1
1.2 Addition and multiplication of real numbers . . . . . 1
1.3 Fields . . . . . 9
1.4 The ordering of the real numbers . . . . . 10
1.5 Order relations and ordered fields . . . . . 18
1.6 The positive integers and the induction principle . . . . . 19
1.7 The integers and the rational numbers . . . . . 24
1.8 The Archimedean postulate; Euclidean completeness . . . . . 26
1.9 The language and notation of sets . . . . . 29
1.10 $n$-fold sums and products; the generalized associative law . . . . . 31
1.11 The closure of the integers under addition and multiplication . . . . . 34

**Chapter 2. INCIDENCE GEOMETRY IN PLANES AND SPACE** . . . . . 36

**Chapter 3. DISTANCE AND CONGRUENCE**

3.1 The idea of a function . . . . . 40
3.2 The set-theoretic interpretation of functions and relations . . . . . 44
3.3 The distance function . . . . . 47
3.4 Betweenness . . . . . 50
3.5 Segments, rays, angles, and triangles . . . . . 54
3.6 Congruence of segments . . . . . 58

**Chapter 4. SEPARATION IN PLANES AND SPACE**

4.1 Convexity and separation . . . . . 61
4.2 Incidence theorems based on the plane-separation postulate . . . . . 64
4.3 Incidence theorems, continued . . . . . 67
4.4 Convex quadrilaterals . . . . . 70
4.5 Separation of space by planes . . . . . 72

**Chapter 5. ANGULAR MEASURE**

5.1 Degree measure for angles . . . . . 74

**Chapter 6. CONGRUENCES BETWEEN TRIANGLES**

6.1 The idea of a congruence . . . . . 81
6.2 The basic congruence theorems . . . . . 85
6.3 Some remarks on terminology . . . . . 90
6.4 The independence of the SAS postulate . . . . . 91
6.5 Existence of perpendiculars . . . . . 93

**Chapter 7. GEOMETRIC INEQUALITIES** . . . . . 95

**Chapter 8. THE EUCLIDEAN PROGRAM: CONGRUENCE WITHOUT DISTANCE**

8.1 The synthetic postulates . . . . . 102
8.2 The laws of inequalities for segments . . . . . 105
8.3 Right angles, synthetically considered . . . . . 108
8.4 The synthetic form of the triangular inequality. Addition of congruence classes . . . . . 111
8.5 A summary of the differences between the metric and synthetic approaches 113

**Chapter 9. THREE GEOMETRIES**

9.1 Introduction . . . . . 114
9.2 The Poincaré model for Lobachevskian geometry . . . . . 115
9.3 The spherical model for Riemannian geometry . . . . . 119
9.4 Some questions for later investigation . . . . . 121

**Chapter 10. ABSOLUTE PLANE GEOMETRY**

10.1 Sufficient conditions for parallelism . . . . . 122
10.2 The polygonal inequality . . . . . 124
10.3 Saccheri quadrilaterals . . . . . 125
10.4 The basic inequality for angle-sums in a triangle . . . . . 128
10.5 A historical comedy . . . . . 130

**Chapter 11. THE PARALLEL POSTULATE AND PARALLEL PROJECTION**

11.1 The uniqueness of parallels . . . . . 132

11.2 Parallel projections . . . . . 134
11.3 The comparison theorem . . . . . 138
11.4 The basic similarity theorem . . . . . 138

**Chapter 12. SIMILARITIES BETWEEN TRIANGLES**

12.1 Proportionalities . . . . . 142
12.2 Similarities between triangles . . . . . 143
12.3 The Pythagorean theorem . . . . . 148

**Chapter 13. POLYGONAL REGIONS AND THEIR AREAS**

13.1 The area postulates . . . . . 153
13.2 Area theorems for triangles and quadrilaterals . . . . . 156
13.3 Applications of area theory: a simple proof of the basic similarity theorem 160
13.4 Further applications of area theory: the Pythagorean theorem . . . . 162
13.5 A weaker form of the unit postulate A–5 . . . . . 165

**Chapter 14. THE CONSTRUCTION OF AN AREA FUNCTION**

14.1 The problem . . . . . 168
14.2 Complexes and their formula areas . . . . . 169
14.3 Verification of the area postulates for the function $\alpha$ . . . . . 174

**Chapter 15. PERPENDICULAR LINES AND PLANES IN SPACE**

15.1 The basic theorems . . . . . 176
15.2 Parallel lines and planes in space . . . . . 181
15.3 The measure of a dihedral angle: perpendicular lines . . . . . 184

**Chapter 16. CIRCLES AND SPHERES**

16.1 Basic definitions . . . . . 186
16.2 Secant and tangent lines. The line-circle theorem . . . . . 187
16.3 Secant and tangent planes . . . . . 190
16.4 Arcs of circles . . . . . 192
16.5 The two-circle theorem . . . . . 197

**Chapter 17. CARTESIAN COORDINATE SYSTEMS**

17.1 Coordinates, equations, and graphs . . . . . 202

**Chapter 18. RIGID MOTION**

18.1 The most general concept of congruence . . . . . 208
18.2 Isometries between triangles . . . . . 211
18.3 General properties of isometries. Reflections . . . . . 213
18.4 Dilations and similarities . . . . . 217

**Chapter 19. CONSTRUCTIONS WITH RULER AND COMPASS**

19.1 Unmarked rulers and collapsible compasses . . . . . 220
19.2 How to do algebra with ruler and compass . . . . . 224
19.3 Solving equations with ruler and compass . . . . . 226
19.4 The problem of Apollonius . . . . . 230
19.5 The impossible construction problems of antiquity . . . . . 233
19.6 The surd field . . . . . 234
19.7 The surd plane . . . . . 236
19.8 Quadratic extension of fields. Conjugates in a quadratic extension field 238
19.9 Surd fields of order $n$; surds of order $n$ . . . . . 241
19.10 Applications to cubic equations with rational coefficients . . . . . 242
19.11 The trisection of the angle . . . . . 244
19.12 The duplication of the cube . . . . . 247

**Chapter 20. FROM EUDOXUS TO DEDEKIND**

20.1 Proportionalities without numbers . . . . . 248
20.2 Eudoxus' synthetic definition of proportionality . . . . . 249
20.3 The algebra of segment addition . . . . . 253
20.4 How to define ratios: the supremum . . . . . 255
20.5 Proof of the addition theorem . . . . . 259
20.6 The metrization theorem . . . . . 264
20.7 The Dedekind cut . . . . . 267

**Chapter 21. LENGTH AND PLANE AREA**

21.1 The definition of arc length . . . . . 269
21.2 The existence of $\pi$ . . . . . 271
21.3 Limits as the mesh approaches zero . . . . . 274
21.4 The addition theorem for arc length . . . . . 276
21.5 Approaching the area of a circular sector from below . . . . . 278
21.6 Approaching the area of a circular sector from above . . . . . 280
21.7 The area of a sector . . . . . 282

**Chapter 22. JORDAN MEASURE IN THE PLANE**

22.1 The basic definition . . . . . 283
22.2 The class of measurable sets . . . . . 286
22.3 Areas under the graphs of continuous functions . . . . . 293

**Chapter 23. SOLID MENSURATION: THE ELEMENTARY THEORY**

23.1 Basic assumptions for the theory of volume . . . . . 297
23.2 Cross sections of cones and pyramids . . . . . 299
23.3 Prisms and cylinders . . . . . 302
23.4 Volumes of prisms and cylinders . . . . . 304
23.5 Volumes of pyramids and cones . . . . . 306
23.6 The volume of a sphere . . . . . 309

**Chapter 24. HYPERBOLIC GEOMETRY**

24.1 Absolute geometry, continued: the critical function . . . . . 312
24.2 Absolute geometry: open triangles and critically parallel rays . . . . . 317
24.3 Hyperbolic geometry: closed triangles and angle sums . . . . . 323
24.4 Hyperbolic geometry: the defect of a triangle and the collapse of similarity theory . . . . . 327
24.5 Absolute geometry: triangulations and subdivisions . . . . . 330
24.6 Euclidean geometry: Bolyai's theorem . . . . . 334
24.7 Hyperbolic area-theory: the defect of a polygonal region . . . . . 337
24.8 Bolyai's theorem for triangles in the hyperbolic case . . . . . 340

24.9 Defects of small triangles . . . . . 343
24.10 The continuity of the defect . . . . . 345
24.11 Bolyai's theorem for polygonal regions in the hyperbolic case . . . . 348
24.12 The impossibility of Euclidean area-theory in hyperbolic geometry . . 349
24.13 The uniqueness of hyperbolic area theory . . . . . 351
24.14 Alternative forms of the parallel postulate . . . . . 352

**Chapter 25. THE CONSISTENCY OF THE HYPERBOLIC POSTULATES**

25.1 Introduction . . . . . 354
25.2 Inversions of a punctured plane . . . . . 354
25.3 Preservation of the cross ratio under inversions . . . . . 358
25.4 Preservation of angular measure under inversions . . . . . 359
25.5 Reflections across L-lines in the Poincaré model . . . . . 363
25.6 Uniqueness of the L-line through two points . . . . . 365
25.7 The ruler postulate, betweenness and plane separation . . . . . 366
25.8 Angular measure in the Poincaré model . . . . . 370
25.9 The SAS postulate . . . . . 370

**Chapter 26. THE CONSISTENCY OF EUCLIDEAN GEOMETRY**

26.1 Introduction . . . . . 373
26.2 The ruler postulate . . . . . 374
26.3 Incidence and parallelism . . . . . 375
26.4 Translations and rotations . . . . . 377
26.5 Plane separation . . . . . 381
26.6 Angle congruences . . . . . 383

**Chapter 27. THE POSTULATIONAL METHOD**

27.1 Introduction . . . . . 388
27.2 Postulates considered as self-evident truths . . . . . 388
27.3 Categoric postulate sets . . . . . 389
27.4 The use of postulate sets as codifications . . . . . 392
27.5 The use of postulates to keep the record straight . . . . . 393

**Chapter 28. AN EXAMPLE OF AN ORDERED FIELD WHICH IS NOT ARCHIMEDEAN** . . . . . 394

**Chapter 29. THE THEORY OF NUMBERS** . . . . . 400

**Chapter 30. THE THEORY OF EQUATIONS** . . . . . 406

**Chapter 31. LIMITS OF SEQUENCES** . . . . . 411

**Chapter 32. COUNTABLE AND UNCOUNTABLE SETS**

32.1 Finite and countable sets . . . . . 415
32.2 The countability of the surd field . . . . . 417
32.3 Proof that real numbers are uncountable . . . . . 418

INDEX . . . . . 421