Contents

	Preface	**xv**
	Symbols	**xix**
Chapter 1	**AFFINE GEOMETRY**	
1	**Intuitive Affine Geometry**	**1**
	Vector space of translations	3
	Limited measurement in the affine plane	5
2	**Axioms for Affine Geometry**	**6**
	Division rings and fields	6
	Axiom system for n-dimensional affine space (X, V, k)	6
	Action of V on X	6
	Dimension of the affine space X	6
	Real affine space $(X, V, \mathbf{R})$	7
3	**A Concrete Model for Affine Space**	**8**
4	**Translations**	**9**
	Definition	9
	Translation group	10
5	**Affine Subspaces**	**11**
	Definition	11
	Dimension of affine subspaces	11
	Lines, planes, and hyperplanes in X	11
	Equality of affine subspaces	14
	Direction space of $S(x, U)$	14
6	**Intersection of Affine Subspaces**	**15**
7	**Coordinates for Affine Subspaces**	**16**
	Coordinate system for V (ordered basis)	16
	Affine coordinate system	17
	Action of V on X in terms of coordinates	18
8	**Analytic Geometry**	**20**
	Parametric equations of a line	20
	Linear equations for hyperplanes	22

9	**Parallelism**	**26**
	Parallel affine subspaces of the same dimension	27
	The fifth parallel axiom	29
	General definition of parallel affine spaces	31
10	**Affine Subspaces Spanned by Points**	**33**
	Independent (dependent) points of X	33
	The affine space spanned by a set of points	35
11	**The Group of Dilations**	**36**
	Definition	37
	Magnifications	38
	The group of magnifications with center c	38
	Classification of dilations	41
	Trace of a dilation	46
12	**The Ratio of a Dilation**	**46**
	Parallel line segments	47
	Line segments, oriented line segments	47
	Ratio of lengths of parallel line segments	47
	Dilation ratios of translations and magnifications	50
	Direction of a translation	54
13	**Dilations in Terms of Coordinates**	**55**
	Dilation ratio	58
14	**The Tangent Space $X(c)$**	**59**
	Definition	59
	Isomorphism between $X(c)$ and $X(b)$	63
	A side remark on high school teaching	64
15	**Affine and Semiaffine Transformations**	**65**
	Semiaffine transformations	65
	The group Sa of semiaffine transformations	71
	Affine transformations	71
	The group Af of affine transformations	72
16	**From Semilinear to Semiaffine**	**73**
	Semilinear mappings	73
	Semilinear automorphisms	75
17	**Parallelograms**	**78**
18	**From Semiaffine to Semilinear**	**81**
	Characterization of semilinear automorphisms of V	84
19	**Semiaffine Transformations of Lines**	**88**

20	**Interrelation among the Groups Acting on X and on V**	**90**
21	**Determination of Affine Transformations by Independent Points and by Coordinates**	**93**
22	**The Theorem of Desargues**	**98**
	The affine part of the theorem of Desargues	99
	Side remark on the projective plane	102
23	**The Theorem of Pappus**	**103**
	Degenerate hexagons	105
	The affine part of the theorem of Pappus	105
	Side remark on associativity	110
	Side remark on the projective plane	110
Chapter 2	**METRIC VECTOR SPACES**	
24	**Inner Products**	**114**
	Definition	114
	Metric vector spaces	114
	Orthogonal (perpendicular) vectors	115
	Orthogonal (perpendicular) subspaces	115
	Nonsingular metric vector spaces	117
25	**Inner Products in Terms of Coordinates**	**119**
	Inner products and symmetric bilinear forms	120
	Inner products and quadratic forms	121
26	**Change of Coordinate System**	**123**
	Congruent matrices	124
	Discriminant of V	125
	Euclidean space	126
	The Lorentz plane	126
	Minkowski space	127
	Negative Euclidean space	128
27	**Isometries**	**131**
	Definition	132
	Remark on terminology	132
	Classification of metric vector spaces	135
28	**Subspaces**	**137**

29	**The Radical**	**141**
	Definition	142
	The quotient space $V/\text{rad } V$	143
	Rank of a metric vector space	143
	Orthogonal sum of subspaces	144
30	**Orthogonality**	**147**
	Orthogonal complement of a subspace	148
	Relationships between U and U^*	149
31	**Rectangular Coordinate Systems**	**153**
	Definition	153
	Orthogonal basis	153
32	**Classification of Spaces over Fields Whose Elements have Square Roots**	**157**
	Orthonormal coordinate system	158
	Orthonormal basis	158
33	**Classification of Spaces over Ordered Fields Whose Positive Elements have Square Roots**	**159**
34	**Sylvester's Theory**	**167**
	Positive semidefinite (definite) spaces	169
	Negative semidefinite (definite) spaces	169
	Maximal positive (negative) definite spaces	170
	Main theorem	172
	Signature of V	172
	Remark about algebraic number fields	177
	Remark about projective geometry	180
35	**Artinian Spaces**	**183**
	Artinian plane	186
	Defense of terminology	186
	Artinian coordinate systems	187
	Properties of Artinian planes	187
	Artinian spaces	189
36	**Nonsingular Completions**	**192**
	Definition	192
	Characterization of Artinian spaces	195
	Orthogonal sum of isometries	197

37	**The Witt Theorem**	**199**
	Fundamental question about isometries	199
	Witt theorem	202
	Witt theorem translated into matrix language	206
38	**Maximal Null Spaces**	**207**
	Definition	207
	Witt index	208
39	**Maximal Artinian Spaces**	**208**
	Definition	209
	Reduction of classification problem to anistropic spaces	211
	A research idea of Artin	211
40	**The Orthogonal Group and the Rotation Group**	**214**
	General linear group $GL(m, k)$	214
	The orthogonal group $O(W)$	215
	Rotations and reflections	217
	180° rotations	218
	Symmetries	219
	Rotation group $O^+(V)$	221
	Remark on teaching high school geometry	222
41	**Computation of Determinants**	**224**
42	**Refinement of the Witt Theorem**	**231**
43	**Rotations of Artinian Space around Maximal Null Spaces**	**238**
44	**Rotations of Artinian Space with a Maximal Null Space as Axis**	**244**
45	**The Cartan–Dieudonné Theorem**	**251**
	Set of generators of a group	251
	Bisector of the vectors A and B	253
	Cartan–Dieudonné theorem	254
46	**Refinement of the Cartan–Dieudonné Theorem**	**257**
	Scherk's theorem	260
47	**Involutions of the General Linear Group**	**262**
48	**Involutions of the Orthogonal Group**	**266**
	Type of an involution	267
	180° rotation	268

49	**Rotations and Reflections in the Plane**	**269**
	Plane reflections	270
	Plane rotations	270
50	**The Plane Rotation Group**	**271**
	Commutativity of $O^+(V)$	272
	Extended geometry from V to V'	273
51	**The Plane Orthogonal Group**	**277**
	The exceptional plane	279
	Characterizations of the exceptional plane	282
52	**Rational Points on Conics**	**282**
	Circle C_r with radius r	283
	Parametric formulas of the circle C_r	287
	Pythagorean triples	288
53	**Plane Trigonometry**	**288**
	Cosine of a rotation	291
	Matrix of a rotation	295
	Orientation of a vector space	297
	Clockwise and counterclockwise rotations	299
	Sine of a rotation	300
	Sum formulas for the sine and cosine	307
	Circle group	308
	Remark on teaching trigonometry	309
54	**Lorentz Transformations**	**311**
55	**Rotations and Reflections in Three-Space**	**315**
	Axis of a rotation	316
	Four classes of isometries	316
	Rotations with a nonsingular line as axis	316
56	**Null Axes in Three-Space**	**318**
57	**Reflections in Three-Space**	**328**
	Reflections which leave only the origin fixed	328
	Two types of reflections	329
	Remark on high school teaching	330
58	**Cartan–Dieudonné Theorem for Rotations**	**331**
	Fundamental question	331
	Cartan–Dieudonné theorem for rotations	332
59	**The Commutator Subgroup of a Group**	**333**

60 The Commutator Subgroup of the Orthogonal Group 337
Birotations 338
Main theorem 338

61 The Commutator Subgroup of the Rotation Group 340

62 The Isometries $\pm 1_V$ 342
Center of a group 342
Magnification with center **0** 343
Magnification and the invariance of lines 343
Isometries which leave all lines through **0** invariant 344

63 Centers of $O(V)$, $O^+(V)$, and $\Omega(V)$ 347
Centralizer in $O(V)$ of the set $O^+(V)^2$ 348
Main theorem 349

64 Linear Representations of the Groups $O(V)$, $O^+(V)$, and $\Omega(V)$ 352
Definition 352
Natural representations of $(O(V), V)$, $(O^+(V), V)$ and $(\Omega(V), V)$ 353
Simple representations 354
Main theorem 356

65 Similarities 358
Definition 358
Main theorem 362
Factorization of similarities into the product of magnifications and isometries 366

Chapter 3 METRIC AFFINE SPACES

66 Square Distance 372
Metric affine space (X, V, k) 372
Orthogonal (perpendicular) affine subspaces 372
Square distance between points 373
The metric vector space $X(c)$ 374
Perpendicular line segments 376
Perpendicular bisector of a line segment 377
Remark on high school teaching 378

67 Rigid Motions 379
Definition and properties 380
n-dimensional Euclidean group 382
Reflections, rotations, symmetries, etc., of X 386
Isometric affine spaces 387

68	**Interrelation among the Groups *Mo*, *Tr*, and $O(V)$**	**387**
	Diagram of relationships	389
	Glide reflections	397
69	**The Cartan–Dieudonné Theorem for Affine Spaces**	**400**
	Parallel symmetries	401
	Cartan–Dieudonné theorem for anisotropic affine spaces	405
	Null motions	407
	Cartan–Dieudonné theorem for affine spaces	407
	Congruent sets	410
70	**Similarities of Affine Spaces**	**412**
	Definition	413
	A similarity as the product of a magnification and rigid motion	415
	Main theorem	418
	Direct and opposite similarities	418
	Angles	420
	Similar figures	424
	Epilogue	**427**
	Bibliography	**429**
	Index	**431**