

## CONTENTS.

### CHAPTER XXIII.

#### CHANGE OF THE VARIABLES IN A MULTIPLE INTEGRAL.

| ARTS.    |  | PAGES |
|----------|--|-------|
| 826-829. | Change of Order of Integration - - - -                             | 1-13  |
| 830-832. | Change of the Variables - - - -                                    | 13-27 |
| 833-840. | Jacobians. BERTRAND'S Definition and Method of Calculation - - - - | 27-33 |
| 841-852. | The General Multiple Integral - - - -                              | 33-43 |
|          | PROBLEMS - - - -   | 44-48 |

### CHAPTER XXIV.

#### EULERIAN INTEGRALS, GAUSS' II FUNCTION.

|          |  |         |
|----------|--|---------|
| 853-870. | Beta and Gamma Functions - - - -   | 49-61   |
| 871-874. | $\int_0^\infty \frac{x^{n-1}}{1+x} dx$ ; $\Gamma(n) \Gamma(1-n) = \pi / \sin n\pi$ ; Gauss' Theorem,<br>a Case - - - - | 61-65   |
| 875-885. | Theorems of WALLIS and STIRLING - - - -  | 65-76   |
| 886-896. | The II Function - - - -  | 76-86   |
| 897-909. | EULER'S Constant. GAUSS' Theorem - - - -   | 87-96   |
| 910-930. | Expansions of $\log F(x+1)$ , etc. Calculation of F Functions. Definite Integrals - - - -                              | 96-116  |
| 931-936. | Approximate Summation - - - -  | 116-120 |
| 937-946. | CAUCHY'S Theorems; Approximation Formulae;<br>DE MORGAN'S Investigation - - - -  | 121-130 |
| 947-954. | Integration by Continued Fractions. Factorial Series for $\psi(a+x)$ ; GAUSS' Calculation of $\psi(x)$ - - - -         | 130-139 |
| 955-957. | TABULATED RESULTS - - - -  | 139-144 |
|          | PROBLEMS - - - -   | 144-152 |

## CHAPTER XXV.

## LEJEUNE-DIRICHLET INTEGRALS, LIOUVILLE INTEGRALS.

| ARTS.    |  | PAGES   |
|----------|--|---------|
| 958-967. | DIRICHLET'S Theorem  | 153-160 |
| 968-984. | Extensions by LIOUVILLE, BOOLE, CATALAN.<br>Generalisations  | 160-173 |
| 985-987. | LIOUVILLE's Integral. LIOUVILLE's Proof of GAUSS'<br>Theorem | 173-175 |
|          | PROBLEMS   | 176-179 |

## CHAPTER XXVI.

## DEFINITE INTEGRALS (I.).

|            |   |         |
|------------|---|---------|
| 988-992.   | General Summary of Methods; $\int_0^{\frac{\pi}{2}} \log \sin x dx$ ; Illustrations   | 180-188 |
| 993-1027.  | Form $\int_0^\infty \frac{\sin^m rx}{x^n} dx$ ( $m < n$ ), etc.   | 188-207 |
| 1028-1030. | WOLSTENHOLME's Principle  | 207-208 |
| 1031-1034. | $\int_0^\infty \frac{\sin^p x}{x^q} dx$ resumed. Partial Fractions  | 209-213 |
| 1035-1041. | $\int_0^\infty \frac{e^{-ax} - e^{-px}}{x} \cos bx dx$ , $\int_0^\infty e^{-x^2} \cos ax dx$ ,<br>$\int_0^\infty e^{-x^2} \frac{\sin ax}{x} dx$ , etc.  | 213-216 |
| 1042-1070. | Various Forms. $\int_0^\infty e^{-c^2(x^2+\frac{a^2}{x^2})} dx$ , $\int_0^\infty \frac{\cos rx}{a^2+x^2} dx$ , etc.;<br>$\int_0^\infty \frac{\sin rx dx}{x(x^4+2a^2x^2\cos 2a+a^4)}$ , $\int_{-\infty}^\infty \frac{\cos rx dx}{(x-b)^2+a^2}$ , etc.;<br>$\int_0^\infty \frac{\cos rx dx}{x^{2n}+a^{2n}}$ , $\int_0^\infty \frac{\cos rx dx}{x^{4n}-2a^{2n}x^{2n}\cos 2na+a^{2n}}$ , etc. | 217-236 |
|            | PROBLEMS  | 237-243 |

## CHAPTER XXVII.

## DEFINITE INTEGRALS (II.).

## LOGARITHMIC AND EXPONENTIAL FUNCTIONS INVOLVED.

|            |  |         |
|------------|--|---------|
| 1071-1076. | Series required  | 244-247 |
| 1077-1085. | Groups A, B, . . . G. Various Forms of Type<br>$\int \frac{x^a \left(\log \frac{1}{x}\right)^p}{(1 \pm x^r)^s} dx$ | 248-259 |

| ARTS.  | PAGES   |
|--|---------|
| 1086-1092. General Principles. $\int_0^\infty \frac{F(x)}{1+x^n} \frac{dx}{x}$ , etc. - - - -            | 260-263 |
| 1093-1096. $\int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta$ , etc. - - - -                          | 263-265 |
| 1097-1098. WOLSTENHOLME'S Expressions for $\int_0^\infty \frac{(\log \frac{1}{x})^{2r+1}}{1 \pm x^n} dx$ | 265-266 |
| 1099-1100. Group H. LEGENDRE'S Rule. KUMMER'S Integrals  | 266-269 |
| 1101-1103. Groups I, J. Derivations from $\int_0^\infty \frac{x^{a-1}}{1 \pm x} dx$ ( $1 > a > 0$ )      | 269-274 |
| 1104-1107. Group K. Type $\int_0^\infty \frac{\cosh px}{\sinh qx} \sin mx dx$ - - - -                    | 274-278 |
| 1108-1120. Groups L, M, N. Formulae of POISSON, ABEL,<br>CAUCHY, LEGENDRE - - - -                        | 278-286 |
| PROBLEMS - - - -   | 287-292 |

## CHAPTER XXVIII.

## DEFINITE INTEGRALS (III.).

|   |         |
|---|---------|
| 1121-1133. Types $\int_0^\pi \cos^p x \cos qx dx$ , etc. ; $\int_0^\pi x^p \sin^{2n} x dx$ , etc.   | 293-301 |
| 1134-1158. Results derivable from well-known Series. Type<br>$\int_0^\pi \frac{\cos p\theta}{(1 - 2\alpha \cos \theta + \alpha^2)^n} d\theta$ - - - - | 301-320 |
| 1159-1168. SERRET's Investigation of $\int_0^\infty e^{-kx} r^{n-1} dx$ , k complex.<br>Deductions - - - -  | 321-327 |
| 1169-1181. FRESNEL's Integrals. SOLDNER's Function li(x) - -  | 327-336 |
| 1182-1188. FRULLANI's Theorem. ELLIOTT's and LEUDES DORF's<br>Extensions - - - -  | 337-342 |
| 1189-1201. Complex Constants - - - -  | 342-352 |
| PROBLEMS - - - -  | 353-361 |

## CHAPTER XXIX.

THE COMPLEX VARIABLE. VECTORS. CONFORMAL  
REPRESENTATION.

|  |         |
|--|---------|
| 1202-1221. Vectors. Laws of Combination. ARGAND Diagram.<br>DEMOIVRE's Theorem - - - - | 362-376 |
| 1222-1234. Continuity, etc. Revision of Definitions - - - -                            | 376-382 |

| ARTS.  |  | PAGES          |
|--|--|----------------|
| 1235-1242. Conformal Representation. WEIERSTRASS' Example<br>of a Function with no determinable Differential<br>Coefficient. Differentiation of a Complex<br>Function - - - - -                    |  | 382-387        |
| 1243-1246. Definitions. Zeros, Singularities, etc. - - - - -   |  | 388-389        |
| 1247-1255. Conformal Representation. Isogonal Property, Con-<br>nection of Elements of Area. Connection of<br>Curvatures. Curvature Formulae. Quasi-Inver-<br>sion. Homographic Relation - - - - - |  | 389-402        |
| 1256-1262. Branch Points. RIEMANN'S Surfaces - - - - -   |  | 402-406        |
| 1263-1265. Number of Roots in a given Contour - - - - -  |  | 407-415        |
| <b>PROBLEMS</b> - - - - -  |  | <b>415-418</b> |

### CHAPTER XXX.

#### INTEGRATION. CONTOUR INTEGRATION. TAYLOR'S THEOREM.

|   |  |                |
|---|--|----------------|
| 1266-1274. Definitions. General Properties. Integration of a<br>Series - - - - -                          |  | 419-423        |
| 1275-1285. CAUCHY'S Theorem. Deformation of a Path. Loops.<br>Exclusion of Singularities - - - - -        |  | 423-434        |
| 1286-1287. $\int \frac{\phi(z) dz}{z-a}$ , $\int \frac{\phi(z) dz}{(z-a_1)(z-a_2)\dots(z-a_r)}$ - - - - - |  | 434-436        |
| 1288-1297. Branch-Points and their Effect - - - - -   |  | 436-446        |
| 1298. Period-Parallelograms - - - - -   |  | 446-447        |
| 1299-1300. Differential Coefficients of a Synectic Function.<br>TAYLOR'S Theorem - - - - -                |  | 448-452        |
| 1301-1328. Contour Integration - - - - -  |  | 452-478        |
| <b>PROBLEMS</b> - - - - -   |  | <b>479-482</b> |

### CHAPTER XXXI.

#### ELLIPTIC INTEGRALS AND FUNCTIONS.

|   |  |         |
|---|--|---------|
| 1329-1342. Periodicity - - - - -  |  | 483-498 |
| 1343. $\operatorname{sn} \iota u$ , $\operatorname{cn} \iota u$ , $\operatorname{dn} \iota u$ - - - - -       |  | 498     |
| 1344-1354. Addition Formulae and Deductions. JACOBI'S 33<br>Formulae. Periodicity - - - - -                   |  | 498-508 |
| 1355-1361. Duplication, Dimidiation, Triplication - - - - -   |  | 508-511 |
| 1362-1365. The General Proposition for Addition Formulae.<br>Second and Third Classes of Legendrian Integrals |  | 511-513 |

| ARTS.   |  | PAGES   |
|---|--|---------|
| 1366-1379. Jacobian Zeta, Eta, Theta Functions. Integrations involving Jacobian Functions - - - - - |  | 514-519 |
| PROBLEMS - - - - -  |  | 520-529 |

## CHAPTER XXXII.

### ELLIPTIC INTEGRALS. THE WEIERSTRASSIAN FORMS.

|   |  |         |
|---|--|---------|
| 1380-1384. Definitions of $\wp(u)$ , $\zeta(u)$ , $\sigma(u)$ . The Differential Coefficients of $\wp(u)$ - - - - - |  | 530-532 |
| 1385-1386. Periodicity of $\wp(u)$ - - - - -  |  | 532-534 |
| 1387-1413. Addition Formula for $\wp(u)$ . Deductions. $\wp(nu) - \wp(u)$ . SCHWARZ - - - - -                       |  | 534-545 |
| 1414-1415. Connection of the Weierstrassian and Legendrian Periods and Functions - - - - -                          |  | 545-547 |
| 1416-1419. Expansions of the Weierstrassian Functions - - - - -   |  | 547-549 |
| 1420-1431. Addition Formulae for Zeta and Sigma Functions and Deductions - - - - -                                  |  | 549-553 |
| 1432-1445. Differentiation of $\wp(u)$ and Integration by Aid of Weierstrassian Functions - - - - -                 |  | 553-560 |
| PROBLEMS - - - - -  |  | 561-566 |

## CHAPTER XXXIII.

|   |  |         |
|---|--|---------|
| ELLIPTIC FUNCTIONS. REDUCTION TO STANDARD FORMS.            |  |         |
| 1446-1458. Reduction to Weierstrassian Form - - - - -       |  | 567-578 |
| 1459-1479. Reduction to Legendrian Form - - - - -           |  | 578-593 |
| 1480-1481. LANDEN'S Transformation. Illustrative Examples - |  | 594-597 |
| PROBLEMS - - - - -  |  | 598-603 |

## CHAPTER XXXIV. (SECTION I.)

### CALCULUS OF VARIATIONS.

|  |  |         |
|--|--|---------|
| 1482-1498. The Symbols $\delta$ , $\omega$ . Notation. Variation of an Integral - - - - -    |  | 604-614 |
| 1499-1502. Conditions for a Stationary Value. The Equation $K = 0$ - - - - -                 |  | 614-620 |
| 1503. Case of $V$ dependent on Terminals - - - - -   |  | 621     |
| 1504-1507. Relative Maxima and Minima. LAGRANGE'S Rule. $V$ a Perfect Differential - - - - - |  | 621-629 |

| ARTS.   | PAGES   |
|---|---------|
| 1508-1523. Several Dependent Variables . . . . .  | 629-639 |
| 1524-1546. Geodesics. Least Action. Brachistochronism. Energy<br>Condition of Equilibrium . . . . . | 639-650 |
| PROBLEMS . . . . .  | 650-660 |

### CHAPTER XXXIV. (SECTION II.)

#### DOUBLE INTEGRALS. CULVERWELL'S METHOD OF DISCRIMINATION.

|  |         |
|--|---------|
| 1547-1557. Double Integrals. Two Independent Variables . . . . .                             | 661-669 |
| 1558-1564. Relative Maxima and Minima. Bubbles. Limited<br>Variation . . . . .               | 669-673 |
| 1565. Stationary Value of $\iint U dS$ , with condition<br>$\iint W dx dy = a$ . . . . .     | 673-674 |
| 1566-1583. Discrimination. CULVERWELL's Method. Conjugate<br>Points. Various Cases . . . . . | 674-691 |
| 1584. BIBLIOGRAPHY . . . . .   | 691     |
| PROBLEMS . . . . .   | 692     |

### CHAPTER XXXV. (SECTION I.)

#### FORMULAE OF LAGRANGE AND FOURIER.

|   |         |
|---|---------|
| 1585-1596. FOURIER's Theorem. The Cosine Series. The Sine<br>Series . . . . . | 693-699 |
| 1597-1604. A Remarkable Limiting Form. POISSON's Investiga-<br>tion . . . . . | 699-710 |
| 1605-1608. GRAPHICAL REPRESENTATION OF THE RESULTS . . . . .                  | 710-712 |
| 1609-1615. REMARKS AND ILLUSTRATIONS . . . . .                                | 712-716 |
| PROBLEMS . . . . .  | 717-719 |

### CHAPTER XXXV. (SECTION II.)

#### DIRICHLET'S INVESTIGATION.

|  |         |
|--|---------|
| 1616-1625. DIRICHLET's Investigation. Existence of Maxima<br>and Minima, Discontinuities . . . . .                                 | 720-728 |
| 1626-1627. Application to FOURIER's Series. CAUCHY's Identity  | 729-730 |
| 1628-1632. The Limit $L_{t \rightarrow \infty} \int_0^h \frac{\sin \omega x}{x} \phi(x) dx$ . Graphical<br>Illustrations . . . . . | 731-734 |
| 1633-1639. Various Deductions. FOURIER's Formula<br>PROBLEMS . . . . .   | 734-736 |
|  | 737-744 |

## CHAPTER XXXVI.

## MEAN VALUES.

| ARTS.   |           | PAGES   |
|---|-----------|---------|
| 1640-1650. General Conception. Combination of Means. Density of a Distribution                      | - - - - - | 745-754 |
| 1651-1654. The Inverse Distance. Theorems on Potential. Typical Examples                            | - - - - - | 754-757 |
| 1655-1656. A Useful Artifice  | - - - - - | 757-766 |
| 1657-1663. $M(\rho^2)$ , $M(\rho^n)$ . Mean Areas and Volumes. Miscellaneous Means                  | - - - - - | 766-778 |
| 1664-1670. Certain Inequalities, and their Consequences. General Means in Terms of Restricted Means | - - - - - | 779-782 |
| 1671-1679. CLERK MAXWELL's "Geometric Mean," useful for Electromagnetic Problems                    | - - - - - | 782-786 |
| PROBLEMS  | - - - - - | 786-791 |

## CHAPTER XXXVII.

## CHANCE.

|   |           |         |
|---|-----------|---------|
| 1680-1687. Definitions and Illustrations of Applications of the Calculus  | - - - - - | 792-800 |
| 1688-1691. Random Points  | - - - - - | 800-802 |
| 1692-1695. Condensation Graph. Illustrations                              | - - - - - | 802-819 |
| 1696-1703. Inverse Probability. BAYES' Problem                            | - - - - - | 820-825 |
| 1704-1706. BUFFON's Problem. Parallel Rulings. Rectangular Rulings        | - - - - - | 825-827 |
| 1707-1711. Random Lines   | - - - - - | 828-830 |
| 1712-1722. Number of Lines crossing a Contour. Various Cases              | - - - - - | 830-834 |
| 1723-1735. Theorems of SYLVESTER and CROFTON                              | - - - - - | 834-844 |
| 1736-1743. D'ALEMBERT's Mortality Curve. Definitions. Expectation of Life | - - - - - | 844-848 |
| PROBLEMS  | - - - - - | 849-852 |

## CHAPTER XXXVIII.

## UNCERTAINTIES OF OBSERVATION.

|   |           |         |
|---|-----------|---------|
| 1744-1749. LAPLACE's Hypothesis. Frequency Law. Weight. Modulus         | - - - - - | 853-857 |
| 1750. Mean Error. Mean of Squares. Error of Mean Square. Probable Error | - - - - - | 857-858 |

| ARTS.  | PAGES   |
|--|---------|
| 1751-1752. KRAMP'S Table. The Graph  | 858-859 |
| 1753-1756. Resultant Weight. Weight of a Series of Observations  | 859-862 |
| 1757-1758. Determination of Error of Mean Square from "Apparent Errors"  | 862-863 |
| 1759-1762. Reduction to Linear Form of Equations connecting Errors of a System of Physical Elements. Standardisation | 863-865 |
| 1763. PRINCIPLE OF LEAST SQUARES   | 865     |
| 1764-1779. The Normal Equations. Order of Procedure.<br>BIBLIOGRAPHY. ILLUSTRATIONS                                  | 866-875 |
| PROBLEMS   | 876-878 |

## CHAPTER XXXIX.

## THEOREMS OF STOKES AND GREEN. HARMONIC ANALYSIS.

|  |         |
|--|---------|
| 1780-1781. STOKES' Theorem   | 879-881 |
| 1782-1784. GREEN'S Theorem. Deductions   | 882-886 |
| 1785-1797. Harmonic Analysis. Spherical, Solid and Surface Harmonics. Poles and Axes   | 886-890 |
| 1798-1808. LEGENDRE's Coefficients. Zonal Harmonics. Properties. Power Series. RODRIGUES' Form. Various Expressions for $P(p)$ | 890-894 |
| 1809-1812. Limiting Values. Expressions in terms of Definite Integrals   | 894-895 |
| 1813-1814. Various Forms of LAPLACE's Equation   | 895-897 |
| 1815-1816. The Differential Equation satisfied by a LEGENDRE's Coefficient. General Solution                                   | 897-898 |
| 1817-1826. $\int_{-1}^1 P_m P_n dp = 0$ , $m \neq n$ . $P_n$ in terms of $p$ . Deductions                                      | 898-902 |
| 1827. $\int_{-1}^1 P_n^2 dp = \frac{2}{2n+1}$  | 902     |
| 1828-1831. $\int_{-1}^1 \frac{P_n}{R^m} dp$ . Calculation of the Coefficient of $P_n$ in $p^n$                                 | 902-905 |
| 1832-1833. Expansion of $f(p)$ in terms of LEGENDRE's Coefficients. Series Unique  | 905     |
| 1834-1835. $P_n'$ , $P_n''$ , etc., in terms of LEGENDRE's Coefficients  | 905-906 |
| 1836-1837. $\int_{-1}^1 P_m' P_n' dp$  | 906-907 |
| 1838-1849. IVORY's Equation. Various Working Theorems  | 907-911 |

| ARTS.  |           | PAGES   |
|--|-----------|---------|
| 1850-1852. Illustrative Applications   | - - - - - | 911-912 |
| 1853. LIST OF WORKING FORMULAE.  | - - - - - | 913     |
| 1854-1856. Roots of $P_n = 0$ . Graphs of $r = aP_n$ , $r = a(1 + \epsilon P_n)$ | - - - - - | 914-917 |
| 1857-1860. $\Sigma(2n+1)P_n$ . A Discontinuity. Physical Meaning                 | - - - - - | 917-920 |
| 1861-1863. Practical Expression of a Rational Function in Solid Harmonics        | - - - - - | 920-921 |
| 1864. Change of Axis   | - - - - - | 921-923 |
| 1865-1867. Tesselar and Sectorial Harmonics                                      | - - - - - | 923-925 |
| 1868-1869. Expansion of $F(\mu, \phi)$ . O'BRIEN'S Proof                         | - - - - - | 925-927 |
| 1870-1871. Integral of Product of two Harmonics over Unit Sphere                 | - - - - - | 927-928 |
| 1872. Theorem as to $V$ within or without the Sphere when known on the Surface   | - - - - - | 928     |
| 1873-1874. Differentiation of Zonal Harmonics. Change of Origin                  | - - - - - | 929-930 |
| PROBLEMS   | - - - - - | 931-939 |

## CHAPTER XL.

## SUPPLEMENTARY NOTES.

|  |           |         |
|--|-----------|---------|
| 1875-1883 A. RIEMANN'S Definition and Theorem  | - - - - - | 940-945 |
| 1884-1889 B. Convergence of an Infinite Integral   | - - - - - | 945-948 |
| 1890 C. Strict Accuracy of Standard Forms  | - - - - - | 948-949 |
| 1891-1893 D. HERMITE'S Method for Rational Fractions   | - - - - - | 949-952 |
| 1894 E. LEGENDRE'S Transformation applied to Form<br>$\int \frac{dx}{X\sqrt{Y}}$   | - - - - - | 952-953 |
| 1895-1899 F. Continuity of a Function of two Independent Variables. Differentiation of a Definite Integral. Double Integrals | - - - - - | 953-956 |
| 1900-1902 G. Uniform Convergence. Double Limits  | - - - - - | 956-957 |
| 1903-1904 H. Unicursal Curves  | - - - - - | 957-959 |
| 1905-1910 I. GENERAL REVIEW  | - - - - - | 959-962 |
| PROBLEMS   | - - - - - | 962-966 |
| ANSWERS  | - - - - - | 966-974 |
| INDEX  | - - - - - | 975-980 |