

CONTENTS

Volume II

CHAPTER 15

SERIES OF NUMBERS

§ 1. <i>Introduction</i>	1
234. Elementary concepts	1
235. The most elementary theorems	3
§ 2. <i>The convergence of positive series</i>	6
236. A condition for the convergence of a positive series	6
237. Theorems on the comparison of series	8
238. Examples	10
239. Cauchy's and d'Alembert's tests	12
240. Raabe's test	15
241. The Maclaurin–Cauchy integral test	18
§ 3. <i>The convergence of arbitrary series</i>	21
242. The principle of convergence	21
243. Absolute convergence	22
244. Alternating series	25
§ 4. <i>The properties of convergent series</i>	27
245. The associative property	27
246. The permuting property of absolutely convergent series	29
247. The case of non-absolutely convergent series	30
248. The multiplication of series	33
§ 5. <i>Infinite products</i>	37
249. Fundamental concepts	37
250. The simplest theorems. The connection with series	38
251. Examples	41
§ 6. <i>The expansion of elementary functions in power series</i>	44
252. Taylor series	44
253. The expansion of the exponential and elementary trigonometrical functions in power series	46

254. Euler's formulae	47
255. The expansion for the inverse tangent	50
256. Logarithmic series	51
257. Stirling's formula	53
258. Binomial series	55
259. A remark on the study of the remainder	57
§ 7. <i>Approximate calculations using series</i>	58
260. Statement of the problem	58
261. The calculation of the number π	60
262. The calculation of logarithms	61

CHAPTER 16

SEQUENCES AND SERIES OF FUNCTIONS

§ 1. <i>Uniform convergence</i>	65
263. Introductory remarks	65
264. Uniform and non-uniform convergence	65
265. The condition for uniform convergence	71
§ 2. <i>The functional properties of the sum of a series</i>	73
266. The continuity of the sum of a series	73
267. The case of positive series	75
268. Termwise transition to a limit	77
269. Termwise integration of series	79
270. Termwise differentiation of series	82
271. An example of a continuous function without a derivative	83
§ 3. <i>Power series and series of polynomials</i>	86
272. The interval of convergence of a power series	86
273. The continuity of the sum of a power series	89
274. Continuity at the end points of the interval of convergence	91
275. Termwise integration of a power series	93
276. Termwise differentiation of a power series	94
277. Power series as Taylor series	96
278. The expansion of a continuous function in a series of polynomials	97
§ 4. <i>An outline of the history of series</i>	100
279. The epoch of Newton and Leibniz	100
280. The period of the formal development of the theory of series	103
281. The creation of a precise theory	107

CHAPTER 17

IMPROPER INTEGRALS

§ 1. <i>Improper integrals with infinite limits</i>	110
282. The definition of integrals with infinite limits	110
283. The application of the fundamental formula of integral calculus	112
284. An analogy with series. Some simple theorems	113
285. The convergence of the integral in the case of a positive function	115
286. The convergence of the integral in the general case	117
287. More refined tests	119
§ 2. <i>Improper integrals of unbounded functions</i>	121
288. The definition of integrals of unbounded functions	121
289. An application of the fundamental formula of integral calculus	124
290. Conditions and tests for the convergence of an integral	125
§ 3. <i>Transformation and evaluation of improper integrals</i>	128
291. Integration by parts in the case of improper integrals	128
292. Change of variables in improper integrals	129
293. The evaluation of integrals by artificial methods	131

CHAPTER 18

INTEGRALS DEPENDING ON A PARAMETER

§ 1. <i>Elementary theory</i>	136
294. Statement of the problem	136
295. Uniform approach to a limit function	137
296. Taking limits under the integral sign	139
297. Differentiation under the integral sign	141
298. Integration under the integral sign	143
299. The case when the limits of the integral also depend on the parameter	144
300. Examples	146
§ 2. <i>Uniform convergence of integrals</i>	147
301. The definition of uniform convergence of integrals	147
302. Conditions and sufficiency tests for uniform convergence	150
303. The case of integrals with finite limits	152
§ 3. <i>The use of the uniform convergence of integrals</i>	154
304. Taking limits under the integral sign	154
305. The integration of an integral with respect to the parameter	157

306. The differentiation of an integral with respect to the parameter	160
307. A remark on integrals with finite limits	161
308. The evaluation of some improper integrals	162
§ 4. <i>Eulerian integrals</i>	168
309. The Eulerian integral of the first type	168
310. The Eulerian integral of the second type	170
311. Some simple properties of the Γ function	171
312. Examples	177
313. Some historical remarks on changing the order of two limit operations	178

CHAPTER 19

IMPLICIT FUNCTIONS. FUNCTIONAL DETERMINANTS

§ 1. <i>Implicit functions</i>	181
314. The concept of an implicit function of one variable	181
315. The existence and properties of an implicit function	183
316. An implicit function of several variables	188
317. The determination of implicit functions from a system of equations	190
318. The evaluation of derivatives of implicit functions	195
§ 2. <i>Some applications of the theory of implicit functions</i>	199
319. Relative extremes	199
320. Lagrange's method of undetermined multipliers	203
321. Examples and problems	204
322. The concept of the independence of functions	207
323. The rank of a functional matrix	209
§ 3. <i>Functional determinants and their formal properties</i>	213
324. Functional determinants	213
325. The multiplication of functional determinants	214
326. The multiplication of non-square functional matrices	216

CHAPTER 20

CURVILINEAR INTEGRALS

§ 1. <i>Curvilinear integrals of the first kind</i>	219
327. The definition of a curvilinear integral of the first kind	219
328. The reduction to an ordinary definite integral	221
329. Examples	224

§ 2. <i>Curvilinear integrals of the second kind</i>	226
330. The definition of curvilinear integrals of the second kind	226
331. The existence and evaluation of a curvilinear integral of the second kind	229
332. The case of a closed contour. The orientation of the plane	232
333. Examples	235
334. The connection between curvilinear integrals of both kinds	237
335. Applications to physical problems	238

CHAPTER 21

DOUBLE INTEGRALS

§ 1. <i>The definition and simplest properties of double integrals</i>	243
336. The problem of the volume of a cylindrical body	243
337. The reduction of a double integral to a repeated integral	245
338. The definition of a double integral	248
339. A condition for the existence of a double integral	249
340. Classes of integrable functions	251
341. The properties of integrable functions and double integrals	254
342. An integral as an additive function of the domain; differentiation in the domain	257
§ 2. <i>The evaluation of a double integral</i>	259
343. The reduction of a double integral to a repeated integral in the case of a rectangular domain	259
344. The reduction of a double integral to a repeated integral in the case of a curvilinear domain	264
345. A mechanical application	271
§ 3. <i>Green's formula</i>	275
346. The derivation of Green's formula	275
347. An expression for area by means of curvilinear integrals	279
§ 4. <i>Conditions for a curvilinear integral to be independent of the path of integration</i>	281
348. The integral along a simple closed contour	281
349. The integral along a curve joining two arbitrary points	282
350. The connection with the problem of exact differentials	285
351. Applications to physical problems	289
§ 5. <i>Change of variables in double integrals</i>	292
352. Transformation of plane domains	292
353. An expression for area in curvilinear coordinates	297
354. Additional remarks	300

355. A geometrical derivation	303
356. Change of variables in double integrals	305
357. The analogy with a simple integral. The integral over an oriented domain	308
358. Examples	309
359. Historical note	312

CHAPTER 22

THE AREA OF A SURFACE. SURFACE INTEGRALS

§ 1. <i>Two-sided surfaces</i>	315
360. Parametric representation of a surface	315
361. The side of a surface	320
362. The orientation of a surface and the choice of a side of it	323
363. The case of a piece-wise smooth surface	326
§ 2. <i>The area of a curved surface</i>	328
364. Schwarz's example	328
365. The area of a surface given by an explicit equation	330
366. The area of a surface in the general case	333
367. Examples	336
§ 3. <i>Surface integrals of the first type</i>	338
368. The definition of a surface integral of the first type	338
369. The reduction to an ordinary double integral	338
370. Mechanical applications of surface integrals of the first type	341
§ 4. <i>Surface integrals of the second type</i>	344
371. The definition of surface integrals of the second type	344
372. The reduction to an ordinary double integral	347
373. Stokes's formula	350
374. The application of Stokes's formula to the investigation of curvilinear integrals in space	354

CHAPTER 23

TRIPLE INTEGRALS

§ 1. <i>A triple integral and its evaluation</i>	357
375. The problem of calculating the mass of a solid	357
376. A triple integral and the conditions for its existence	358
377. The properties of integrable functions and triple integrals	359
378. The evaluation of a triple integral	361
379. Mechanical applications	365

§ 2. <i>Ostrogradski's formula</i>	368
380. Ostrogradski's formula	368
381. Some examples of applications of Ostrogradski's formula	371
§ 3. <i>Change of variables in triple integrals</i>	375
382. The transformation of space domains	375
383. An expression for volume in curvilinear coordinates	377
384. A geometrical derivation	380
385. Change of variables in triple integrals	383
386. Examples	384
387. Historical note	387
§ 4. <i>The elementary theory of a field</i>	388
388. Scalars and vectors	388
389. Scalar and vector fields	389
390. A derivative in a given direction. Gradient	390
391. The flow of a vector through a surface	393
392. Ostrogradski's formula. Divergence	394
393. The circulation of a vector. Stokes's formula. Vortex	397
§ 5. <i>Multiple integrals</i>	400
394. The volume of an m -dimensional body and the m -tuple integral	400
395. Examples	401

CHAPTER 24

FOURIER SERIES

§ 1. <i>Introduction</i>	404
396. Periodic values and harmonic analysis	404
397. The determination of coefficients by the Euler-Fourier method	407
398. Orthogonal systems of functions	410
§ 2. <i>The expansion of functions in Fourier series</i>	412
399. Statement of the problem. Dirichlet's integral	412
400. A fundamental lemma	415
401. The principle of localization	417
402. The representation of a function by Fourier series	418
403. The case of a non-periodic function	421
404. The case of an arbitrary interval	422
405. An expansion in cosines only, or in sines only	424
406. Examples	427

407. The expansion of a continuous function in a series of trigonometrical polynomials	433
§ 3. <i>The Fourier integral</i>	435
408. The Fourier integral as a limiting case of a Fourier series	435
409. Preliminary remarks	437
410. The representation of a function by a Fourier integral	439
411. Different forms of Fourier's formula	440
412. Fourier transforms	442
§ 4. <i>The closed and complete nature of a trigonometrical system of functions</i>	445
413. Mean approximation to functions. Extreme properties of a Fourier series	445
414. The closure of a trigonometrical system	448
415. The completeness of a trigonometrical system	453
416. The generalized equation of closure	454
417. Termwise integration of a Fourier series	455
418. The geometrical interpretation	456
§ 5. <i>An outline of the history of trigonometrical series</i>	462
419. The problem of the vibration of a string	462
420. D'Alembert's and Euler's solution	463
421. Taylor's and D. Bernoulli's solution	465
422. The controversy concerning the problem of the vibration of a string	468
423. The expansion of functions in trigonometrical series; the determination of coefficients	470
424. The proof of the convergence of Fourier series and other problems	472
425. Concluding remarks	474

CONCLUSION

AN OUTLINE OF FURTHER DEVELOPMENTS IN MATHEMATICAL ANALYSIS

I. The theory of differential equations	476
II. Variational calculus	479
III. The theory of functions of a complex variable	484
IV. The theory of integral equations	488
V. The theory of functions of a real variable	492
VI. Functional analysis	498

<i>Index</i>	507
--------------	-----

<i>Other Titles in the Series</i>	517
-----------------------------------	-----

*Volume I**Foreword*

xxiii

CHAPTER 1

REAL NUMBERS

§ 1. <i>The set of real numbers and its ordering</i>	1
1. Introductory remarks	1
2. Definition of irrational number	2
3. Ordering of the set of real numbers	5
4. Representation of a real number by an infinite decimal fraction	7
5. Continuity of the set of real numbers	10
6. Bounds of number sets	11
§ 2. <i>Arithmetical operations over real numbers</i>	14
7. Definition and properties of a sum of real numbers	14
8. Symmetric numbers. Absolute quantity	15
9. Definition and properties of a product of real numbers	17
§ 3. <i>Further properties and applications of real numbers</i>	18
10. Existence of a root. Power with a rational exponent	18
11. Power with an arbitrary real exponent	20
12. Logarithms	21
13. Measuring segments	22

CHAPTER 2

FUNCTIONS OF ONE VARIABLE

§ 1. <i>The concept of a function</i>	25
14. Variable quantity	25
15. The domain of variation of a variable quantity	26
16. Functional relation between variables. Examples	27
17. Definition of the concept of function	28
18. Analytic method of prescribing a function	31
19. Graph of a function	33
20. Functions of positive integral argument	36
21. Historical remarks	37
§ 2. <i>Important classes of functions</i>	39
22. Elementary functions	39
23. The concept of the inverse function	44
24. Inverse trigonometric functions	46
25. Superposition of functions. Concluding remarks	50

THEORY OF LIMITS

§ 1. <i>The limit of a function</i>	52
26. Historical remarks	52
27. Numerical sequence	52
28. Definition of the limit of a sequence	54
29. Infinitesimal quantities	56
30. Examples	57
31. Infinitely large quantities	60
32. Definition of the limit of a function	61
33. Another definition of the limit of a function	63
34. Examples	65
35. One-sided limits	71
§ 2. <i>Theorems on limits</i>	72
36. Properties of functions of a positive integral argument, possessing a finite limit	72
37. Extension to the case of a function of an arbitrary variable	74
38. Passage to the limit in equalities and inequalities	75
39. Theorems on infinitesimals	77
40. Arithmetical operations on variables	79
41. Indefinite expressions	80
42. Extension to the case of a function of an arbitrary variable	83
43. Examples	84
§ 3. <i>Monotonic functions</i>	88
44. Limit of a monotonic function of a positive integral argument	88
45. Examples	91
46. A lemma on imbedded intervals	92
47. The limit of a monotonic function in the general case	93
§ 4. <i>The number e</i>	95
48. The number e defined as the limit of a sequence	95
49. Approximate computation of the number e	97
50. The basic formula for the number e . Natural logarithms	99
§ 5. <i>The principle of convergence</i>	102
51. Partial sequences	102
52. The condition of existence of a finite limit for a function of positive integral argument	104
53. The condition of existence of a finite limit for a function of an arbitrary argument	106

§ 6. <i>Classification of infinitely small and infinitely large quantities</i>	108
54. Comparison of infinitesimals	108
55. The scale of infinitesimals	109
56. Equivalent infinitesimals	110
57. Separation of the principal part	111
58. Problems	112
59. Classification of infinitely large quantities	114

CHAPTER 4

CONTINUOUS FUNCTIONS OF ONE VARIABLE

§ 1. <i>Continuity (and discontinuity) of a function</i>	115
60. Definition of the continuity of a function at a point	115
61. Condition of continuity of a monotonic function	117
62. Arithmetical operations over continuous functions	119
63. Continuity of elementary functions	119
64. The superposition of continuous functions	121
65. Computation of certain limits	122
66. Power-exponential expressions	124
67. Classification of discontinuities. Examples	125
§ 2. <i>Properties of continuous functions</i>	127
68. Theorem on the zeros of a function	127
69. Application to the solution of equations	129
70. Mean value theorem	130
71. The existence of inverse functions	132
72. Theorem on the boundedness of a function	133
73. The greatest and smallest values of a function	134
74. The concept of uniform continuity	136
75. Theorem on uniform continuity	138

CHAPTER 5

DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE

§ 1. <i>Derivative of a function and its computation</i>	140
76. Problem of calculating the velocity of a moving point	140
77. Problem of constructing a tangent to a curve	142
78. Definition of the derivative	145
79. Examples of the calculation of the derivative	149

80. Derivative of the inverse function	151
81. Summary of formulae for derivatives	154
82. Formula for the increment of a function	154
83. Rules for the calculation of derivatives	156
84. Derivative of a compound function	158
85. Examples	159
86. One-sided derivatives	161
87. Infinite derivatives	162
88. Further examples of exceptional cases	164
§ 2. <i>The differential</i>	165
89. Definition of the differential	165
90. The relation between the differentiability and the existence of the derivative	166
91. Fundamental formulae and rules of differentiation	168
92. Invariance of the form of the differential	170
93. Differentials as a source of approximate formulae	171
94. Application of differentials in estimating errors	172
§ 3. <i>Derivatives and differentials of higher orders</i>	173
95. Definition of derivatives of higher orders	173
96. General formulae for derivatives of arbitrary order	175
97. The Leibniz formula	177
98. Differentials of higher orders	180
99. Violation of the invariance of the form for differentials of higher orders	181

CHAPTER 6

BASIC THEOREMS OF DIFFERENTIAL CALCULUS

§ 1. <i>Mean value theorems</i>	183
100. Fermat's theorem	183
101. Rolle's theorem	185
102. Theorem on finite increments	186
103. The limit of the derivative	189
104. Generalized theorem on finite increments	189
§ 2. <i>Taylor's formula</i>	191
105. Taylor's formula for a polynomial	191
106. Expansion of an arbitrary function	193
107. Another form for the remainder term	196
108. Application of the derived formulae to elementary functions	199
109. Approximate formulae. Examples	201

CHAPTER 7

INVESTIGATION OF FUNCTIONS BY MEANS OF DERIVATIVES

§ 1. <i>Investigation of the behaviour of functions</i>	204
110. Conditions that a function may be constant	204
111. Condition of monotonicity of a function	205
112. Maxima and minima; necessary conditions	207
113. The first rule	209
114. The second rule	211
115. Construction of the graph of a function	212
116. Examples	214
117. Application of higher derivatives	216
§ 2. <i>The greatest and the smallest values of a function</i>	218
118. Determination of the greatest and the smallest values	218
119. Problems	219
§ 3. <i>Solution of indeterminate forms</i>	221
120. Indeterminate forms of the type $0/0$	221
121. Indeterminate forms of the type ∞/∞	224
122. Other types of indeterminate forms	227

CHAPTER 8

FUNCTIONS OF SEVERAL VARIABLES

§ 1. <i>Basic concepts</i>	229
123. Functional dependence between variables. Examples	229
124. Functions of two variables and their domains of definition	230
125. Arithmetic m -dimensional space	233
126. Examples of domains in m -dimensional space	236
127. General definition of open and closed domains	238
128. Function of m variables	240
129. Limit of a function of several variables	242
130. Examples	245
131. Repeated limits	246
§ 2. <i>Continuous functions</i>	249
132. Continuity and discontinuities of functions of several variables	249
133. Operations on continuous functions	251
134. Theorem on the vanishing of a function	252
135. The Bolzano–Weierstrass lemma	253
136. Theorem on the boundedness of a function	254
137. Uniform continuity	255

CHAPTER 9

DIFFERENTIATION OF FUNCTIONS OF SEVERAL VARIABLES

§ 1. <i>Derivatives and differentials of functions of several variables</i>	258
138. Partial derivatives	258
139. Total increment of the function	260
140. Derivatives of compound functions	263
141. Examples	265
142. The total differential	266
143. Invariance of the form of the (first) differential	268
144. Application of the total differential to approximate calculations	270
145. Homogeneous functions	272
§ 2. <i>Derivatives and differentials of higher orders</i>	275
146. Derivatives of higher orders	275
147. Theorems on mixed derivatives	277
148. Differentials of higher orders	280
149. Differentials of compound functions	283
150. The Taylor formula	284
§ 3. <i>Extrema, the greatest and the smallest values</i>	286
151. Extrema of functions of several variables. Necessary conditions	286
152. Investigation of stationary points (for the case of two variables)	288
153. The smallest and the greatest values of a function. Examples	292
154. Problems	295

CHAPTER 10

PRIMITIVE FUNCTION (INDEFINITE INTEGRAL)

§ 1. <i>Indefinite integral and simple methods for its evaluation</i>	299
155. The concept of a primitive function (and of an indefinite integral)	299
156. The integral and the problem of determination of area	302
157. Collection of the basic integrals	305
158. Rules of integration	306
159. Examples	308
160. Integration by a change of variable	309
161. Examples	312
162. Integration by parts	314
163. Examples	315
§ 2. <i>Integration of rational expressions</i>	318
164. Formulation of the problem of integration in finite form	318
165. Simple fractions and their integration	319

166. Integration of proper fractions	321
167. Ostrogradski's method for separating the rational part of an integral	324
§ 3. <i>Integration of some expressions containing roots</i>	327
168. Integration of expressions of the form $R\left[x, \sqrt[m]{\frac{\alpha x + \beta}{\gamma x + \delta}}\right]$	327
169. Integration of binomial differentials	329
170. Integration of expressions of the form $R[x, \sqrt{(ax^2 + bx + c)}]$. Euler's substitution	331
§ 4. <i>Integration of expressions containing trigonometric and exponential functions</i>	
171. Integration of the differentials $R(\sin x, \cos x)dx$	336
172. Survey of other cases	339
§ 5. <i>Elliptic integrals</i>	341
173. Definitions	341
174. Reduction to the canonical form	341

CHAPTER 11

DEFINITE INTEGRAL

§ 1. <i>Definition and conditions for the existence of a definite integral</i>	344
175. Another formulation of the area problem	344
176. Definition	346
177. Darboux's sums	348
178. Condition for the existence of the integral	350
179. Classes of integrable functions	352
§ 2. <i>Properties of definite integrals</i>	354
180. Integrals over an oriented interval	354
181. Properties expressed by equalities	356
182. Properties expressed by inequalities	357
183. Definite integral as a function of the upper limit	361
§ 3. <i>Evaluation and transformation of definite integrals</i>	364
184. Evaluation using integral sums	364
185. The fundamental formula of integral calculus	365
186. The formula for the change of variable in a definite integral	367
187. Integration by parts in a definite integral	368
188. Wallis's formula	370
§ 4. <i>Approximate evaluation of integrals</i>	371
189. The trapezium formula	371
190. Parabolic formula	374
191. Remainder term for the approximate formulae	376
192. Example	379

CHAPTER 12

GEOMETRIC AND MECHANICAL APPLICATIONS OF THE
INTEGRAL CALCULUS

§ 1. <i>Areas and volumes</i>	381
193. Definition of the concept of area. Quadrable domains	381
194. The additive property of area	384
195. Area as a limit	385
196. An integral expression for area	385
197. Definition of the concept of volume and its properties	390
198. Integral expression for the volume	392
§ 2. <i>Length of arc</i>	399
199. Definition of the concept of the length of an arc	399
200. Lemmas	401
201. Integral expression for the length of an arc	402
202. Variable arc and its differential	406
203. Length of the arc of a spatial curve	408
§ 3. <i>Computation of mechanical and physical quantities</i>	409
204. Applications of definite integrals	409
205. The area of a surface of revolution	412
206. Calculation of static moments and centre of mass of a curve	415
207. Determination of static moments and centre of mass of a plane figure	418
208. Mechanical work	420

CHAPTER 13

SOME GEOMETRIC APPLICATIONS OF THE DIFFERENTIAL
CALCULUS

§ 1. <i>The tangent and the tangent plane</i>	423
209. Analytic representation of plane curves	423
210. Tangent to a plane curve	425
211. Positive direction of the tangent	430
212. The case of a spatial curve	432
213. The tangent plane to a surface	435
§ 2. <i>Curvature of a plane curve</i>	438
214. The direction of concavity, points of inflection	438
215. The concept of curvature	440
216. The circle of curvature and radius of curvature	444

CHAPTER 14

HISTORICAL SURVEY OF THE DEVELOPMENT OF THE
FUNDAMENTAL CONCEPTS OF MATHEMATICAL ANALYSIS

§ 1. <i>Early history of the differential and integral calculus</i>	448
217. Seventeenth century and the analysis of infinitesimals	448
218. The method of indivisibles	449
219. Further development of the science of indivisibles	452
220. Determination of the greatest and smallest quantities; construction of tangents	455
221. Construction of tangents by means of kinematic considerations	458
222. Mutual invertibility of the problems of construction of tangent and squaring	460
223. Survey of the foregoing achievements	462
§ 2. <i>Isaac Newton (1642-1727)</i>	463
224. The calculus of fluxions	463
225. The calculus inverse to the calculus of fluxions; squaring	467
226. Newton's <i>Principles</i> and the origin of the theory of limits	471
227. Problems of foundations in Newton's works	472
§ 3. <i>Gottfried Wilhelm Leibniz (1646-1716)</i>	473
228. First steps in creating the new calculus	473
229. The first published work on differential calculus	475
230. The first published paper on integral calculus	477
231. Further works of Leibniz. Creation of a school	479
232. Problems of foundation in Leibniz's works	480
233. Postscript	481
<i>Index</i>	483
<i>Other Titles in the Series</i>	493