CONTENTS

From the preface to the Hungarian edition 17
Foreword to the English edition 21
Historical overview 23

PART ONE

INTRODUCTION TO THE THEORY OF REAL FUNCTIONS

CHAPTER 1: ELEMENTS OF SET THEORY

1.1 Set relations, operations with sets and characteristic functions 27
1.11 Fundamental notions – 1.12 Algebra of sets – 1.13 Sequences of sets – 1.14 Characteristic functions of sets

1.2 Countable and non-countable sets 30
1.21 The concept of countability – 1.22 Examples – 1.23 The non-countability of continua – 1.24 Additional observations

1.3 Comparison of cardinals and theorem of equivalence 34
1.31 Cardinality and equivalence of sets – 1.32 The Cantor–Bernstein theorem

1.4 Sets of higher cardinality, transfinite cardinals 36
1.41 Cardinality of Euclidean spaces – 1.42 Sets of higher cardinality than the continuum power – 1.43 The notion of power-set – 1.44 Transfinite induction and cardinals

1.5 Important set structures: Boolean and sigma-rings 40
1.51 Fields of sets: Boolean ring (algebra) and σ-ring (algebra) – 1.52 Generating fields of sets, Borel sets

1.6 Simple applications in elementary analysis 42
1.61 Countability of discontinuities of the first kind – 1.62 Countability of break points – 1.63 Cardinal of the set of functions continuous in an interval

CHAPTER 2: POINT SETS IN EUCLIDEAN SPACES

2.1 Distance and neighbourhood in space E_ν 47

2.11 Distance of points in E_ν, fundamental inequalities – 2.12 Neighbourhood of a point and some ideas connected with that – 2.13 Topological mapping, foundations of the general theory of curves and surfaces, C. Jordan's curve theorem

2.2 Convergence and clustering point of a sequence of points 50

2.21 Fundamentals on convergent sequences of points, Cauchy's criterion – 2.22 Clustering point, upper and lower limit – 2.23 Connection between the notions of convergence and clustering point

2.3 Notions and theorems concerning derivate sets 53

2.31 Accumulation point of a point set, the Bolzano–Weierstrass theorem, compactness – 2.32 Condensation point, Lindelöf's theorem – 2.33 Derivate of a point set and related notions – 2.34 Closure of a set – 2.35 Perfectness of the set of condensation points – 2.36 On the cardinality of sets consisting of accumulation and condensation points

2.4 Basic properties of closed and open sets 57

2.41 Closure and boundary points – 2.42 Openness and related concepts – 2.43 Principle of duality; union and meet of closed and open sets – 2.44 The normal representation of a linear open set – 2.45 Cantor's ternary set

2.5 The Cantor–Bendixson theorem, Cantor's meet theorem 62

2.51 Distance of point sets – 2.52 The theorem of Cantor and Bendixson – 2.53 Diameter of a point set and the meet theorem of Cantor

2.6 Covering theorems, outer measures, null sets 65

2.61 Borel's covering theorem – 2.62 Jordan's and Lebesgue's outer measure – 2.63 Null sets – 2.64 Elementary properties of outer measures – 2.65 Sierpinski's lemma and Vitali's covering theorem

CHAPTER 3: EXTENSION OF THE CONCEPT OF FUNCTION. LIMIT, CONTINUITY AND DIFFERENTIABILITY OF A POINT FUNCTION

3.1 Functions on abstract sets (operators) and their classification 73

3.11 The general concept of function, Cantor's theorem on the cardinality of operator sets – 3.12 Classification of operators; linearity, boundedness and norm of a functional (functional operation) – 3.13 The notion of linear operator

3.2 Limit and cluster value of a point function, relating to a set 76

3.21 The limit of a point function at an accumulation point of the basic set – 3.22 Cluster value (sublimit) of a point function, upper and lower limit – 3.23 Upper and lower fluctuation value – 3.24 The analogue of Cauchy's criterion, operations with generalized limits

3.3 Properties of continuous and semi-continuous point functions 80

3.31 Local continuity and semi-continuity relating to a basic set – 3.32 Characterization of point functions continuous on a set, closure with respect to the fundamental operations – 3.33 Main properties: extension of theorems of Weierstrass, Bolzano and Heine

3.4 The approximation theorems of Weierstrass; Stone's theorem 83

3.41 The two approximation theorems of Weierstrass – 3.42 Uniform approximation on a prescribed set by elements of some class of functions, Stone's theorem – 3.43 Conclusions and remarks

3.5 Sequences and series of continuous point functions 88

3.51 Uniform convergence on some set of sequence or series of point functions, the Weierstrass criterion – 3.52 Use of uniform convergence in case of continuous functions, quasi-uniform convergence – 3.53 Baire's and Young's classes of functions

3.6 Generalized differentiability, derivate numbers 91

3.61 Total and partial differential, differential coefficients of a point function – 3.62 Generalized derivate numbers – 3.63 Geometric interpretation, main properties of differentiable functions

3.7 Constructing everywhere continuous, nowhere differentiable functions 94

3.71 Historical remarks, example of Weierstrass – 3.72 An elementary example of a function continuous everywhere whose derivative does not exist at any point – 3.73 Extension of the latter construction, further developments

CHAPTER 4: MONOTONIC, ABSOLUTELY CONTINUOUS FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

4.1 Basic properties, continuous and pure jumping part of a monotonic function 99

4.11 Monotony and the existence of one-sided limits – 4.12 Constructing a pure saltus function with prescribed jumps – 4.13 Decomposition of an arbitrary bounded monotonic function into continuous and pure jumping part – 4.14 Continuity almost everywhere of a monotonic function

4.2 Differentiability almost everywhere of monotonic functions 103

4.21 Lebesgue's theorem on monotonic functions (Proof after Rajchman and Saks) – 4.22 Historical comments

4.3 Applications; Fubini's theorem for series of functions 106

4.31 The Denjoy–Saks–Young theorem on derivate numbers – 4.32 Theorem of Fubini on term-by-term differentiation of a series of monotonic functions

4.4 Functions of bounded variation, the notion of total variation 108

4.41 Rectifiable curves and functions of bounded variation; the total variation – 4.42 Some elementary observations, additivity of total variation – 4.43 Closure of the class of functions of bounded variation with respect to the fundamental operations

4.5 Jordan's and Lebesgue's theorems on functions of bounded variation .. 111

4.51 Jordan's theorem: representation of a function of bounded variation by means of monotonic functions – 4.52 Consequences, the theorem of Lebesgue – 4.53 Example of a monotone increasing and continuous, non-constant function with derivative vanishing almost everywhere

4.6 Absolutely continuous functions 113
4.61 The idea of absolutely continuous function as a tightening of the notion of function of bounded variation – 4.62 Connection with uniform continuity and differentiability; ordinary Lipschitz condition – 4.63 Jordan's theorem for absolutely continuous functions – 4.64 The constancy of an absolutely continuous function with derivative vanishing almost everywhere – 4.65 Extension of the results of this chapter for point functions of several variables; utilization of interval functions

CHAPTER 5: JORDAN MEASURE AND RIEMANN INTEGRAL. RIEMANN–STIELTJES INTEGRAL

5.1 Foundations of Jordan's measure theory 119
5.11 Fundamental concepts of the general measure theory – 5.12 Jordan measurability and Jordan measure in space E_ν – 5.13 Basic properties of $\mu_J(S)$

5.2 Riemann integral of a function of one variable as Jordan measure with sign; upper and lower Darboux integral 122
5.21 Connection between Riemann's integral concept and Jordan's theory of measure – 5.22 Darboux integrals and limit definition of the Riemann integral

5.3 Riemann's and Lebesgue's criteria for (R)-integrability 125
5.31 Oscillation sum and Riemann integral – 5.32 Lebesgue's criterion: continuity almost everywhere of a Riemann integrable function

5.4 Deeper study of (R)-integrable functions; limiting process under the (R)-integral sign 128
5.41 The role of the class of (R)-integrable functions in elementary analysis – 5.42 Operations with (R)-integrable functions; problem of passage to the limit term-by-term – 5.43 Example of a sequence (or series) of (R)-integrable functions whose limit (sum) is not (R)-integrable

5.5 Use of a primitive; properties of the indefinite (R)-integral 131
5.51 The Leibniz–Newton formula for an arbitrary (R)-integrable function – 5.52 Investigation of the existence of a primitive – 5.53 Properties of (R)-integral functions – 5.54 Connection between the operations of (R)-integration and differentiation – 5.55 Volterra's example of a function differentiable in some closed interval whose derivative is bounded but not (R)-integrable

5.6 Improper (R)-integral and Riemann integral of several dimensions 136
5.61 Kinds of improper (R)-integrals – 5.62 (R)-integral on a region in the space E_ν – 5.63 Iterated (R)-integration, extended integral mean

5.7 The Stieltjesian generalization of Riemann integral 138
5.71 Mathematical and physical background of the extension due to Stieltjes – 5.72 Definition of the (RS)-integral; calculation of $\int_a^b f(x)\,dg(x)$ if $f(x)$ is continuous and $g(x)$ is a step function – 5.73 Additivity, integration by parts for (RS)-integrals – 5.74 Conditions for existence of $\int_a^b f\,dg$; the cases when the integrator function g is monotonic, of bounded variation or absolutely continuous – 5.75 Main properties of the (RS)-integral, mean value theorems

5.8 Connection of (RS)-integrals with (R)-integrals; Radon's and Burkill's integral 145

5.81 Transforming $\int_a^b f(x)\,dg(x)$ into an (R)-integral if $g'(x)$ exist and is (R)-integrable – 5.82 (RS)-integral of several dimensions (Radon's integral), derivative with respect to a region and the notion of density – 5.83 Burkill integral of an interval function as a generalized (RS)-integral

CHAPTER 6: LEBESGUE'S MEASURE AND INTEGRAL. MEASURABLE FUNCTIONS

6.1 Elements of Lebesgue's measure theory 149

6.11 Defects of Riemann's concept of integral, the need of its developing – 6.12 Lebesgue measurable sets. Lebesgue measure in the space E_ν – 6.13 Basic properties of $\mu_L(S)$ – 6.14 Total additivity of the (L)-measure

6.2 Geometric introduction of the Lebesgue integral of a function of one variable; upper and lower Young integrals 154

6.21 (L)-integral of a non-negative function as the (L)-measure of an ordinate set, Young integrals – 6.22 Extension to bounded function of arbitrary sign –6.23 Some elementary features of the (L)-integral

6.3 Lebesgue's original definition for the (L)-integral of a bounded function; (L)-integrability and measurability 157

6.31 Definition of the (L)-integral by means of Lebesgue sums, the connection of the latter with Young sums – 6.32 The notion of measurable function, level sets – 6.33 Equivalence of the (L)-integrability and measurability of a bounded function

6.4 Other definition forms; fundamental idea of the treatment due to F. Riesz 163

6.41 Limit definition of the (L)-integral – 6.42 Use of (L)-integrals formed from characteristic functions – 6.43 The Riesz theory of (L)-integral

6.5 Comparison of Lebesgue's concept of integral with that of Riemann; properties of measurable functions 165

6.51 (L)-integrability and (R)-integrability – 6.52 Basic properties of measurable functions – 6.53 Closure with respect to the limit operation

6.6 Almost uniform approximation of a measurable function by measurable functions; relationship between measurability and continuity 168

6.61 Almost uniform convergence, Egorov's theorem – 6.62 Lusin's theorem – 6.63 "Correction" of a measurable function to a continuous one

CHAPTER 7: PROPERTIES OF THE LEBESGUE INTEGRAL. GENERALIZED LEBESGUE INTEGRALS

7.1 Fundamental properties of the (L)-integral of a bounded function over a measurable set 173

7.11 (L)-integral taken on a set – 7.12 (L)-equivalence, basic properties, mean value theorems – 7.13 Absolute (total) continuity of the (L)-integral – 7.14 Total additivity of the (L)-integral

7.2 Integration of sequences and series of functions 176

7.21 The limit theorem of Lebesgue – 7.22 Convergence with respect to (L)-measure – 7.23 Term-by-term integration of a series consisting of measurable functions and Arzelà's theorem

7.3 Indefinite (L)-integral with bounded integrand and the notion of primitive 179

7.31 Calculation of an (L)-integral by means of a primitive – 7.32 Determination of primitives for a bounded function by (L)-integration – 7.33 Properties of (L)-integral functions, notion of Lebesgue point – 7.34 The role of Lischitz condition in the theory, relation between (L)-integration and differentiation

7.4 (L)-integral with unbounded basic set or integrand 183

7.41 Extension of Lebesgue's integral definition, generalized Lebesgue sum – 7.42 Reduction of the (L)-integral of an unbounded function to the case of bounded integrand – 7.43 Remarks on (L)-integrability in wider sense, the class $L(S)$

7.5 Passage to the limit under the sign of a generalized (L)-integral 187

7.51 Lebesgue's main theorem on integration of a sequence of functions – 7.52 Fatou's lemma and the theorem of B. Levi – 7.53 A pendant of uniform continuity in the class $L(S)$

7.6 Theorems on arbitrary (L)-integral functions 194

7.61 Extension of the Leibniz–Newton formula – 7.62 (L)-integrating the derivative of a function of bounded variation – 7.63 Absolute continuity of an arbitrary (L)-integral function – 7.64 Derivation of the indefinite (L)-integral and Lebesgue points – 7.65 Absolute continuity as a characteristic property, the total variation of an (L)-integral function – 7.66 (L)-integral function of an unbounded derivative, extensions of the (L)-integral due to Denjoy and Perron – 7.67 Integration by parts and by substitution for (L)-integrals

7.7 (L)-integral of several dimensions; Fubini's theorem on successive integration 203

7.71 (L)-integral of a point function of several variables, Fubini's theorem for double (L)-integrals – 7.72 Remarks on the generalization of Lebesgue's theory.

CHAPTER 8: MEASURES DEFINED ON BOOLEAN RINGS AND THE ABSTRACT LEBESGUE INTEGRAL. LEBESGUE–STIELTJES INTEGRAL

8.1 Extension of measures, Carathéodory's theorem 205

8.11 Introduction – 8.12 Measure and measure space defined on a Boolean ring – 8.13 The notion of completeness and Borel's measure; closure of a measure space –8.14 Generating an abstract measure by means of an outer measure induced by premeasure – 8.15 Carathéodory's extension theorem, Lebesgue's continuation of premeasure

8.2 Lebesgue integral on abstract measure space 208

8.21 Measurability of a function on abstract space, analogues of the abstract Lebesgue integral – 8.22 Definition of the abstract Lebesgue integral – 8.23 Properties of abstract (L)-integrals, Daniell integral

8.3 Lebesgue–Stieltjes measure and integral 211

8.31 The Lebesgue–Stieltjes measure as a particular case of the notion of abstract measure – 8.32 (LS)-integral and its connection with the (RS)-integral – 8.33 Relation between (LS)-integral and (L)-integral – 8.34 Integration by parts and transformation of (LS)-integrals – 8.35 Development of the theory of (LS)-integral, applications in probability theory

8.4 Measures and integrals on product spaces 217

8.41 (L)-measures of higher dimension generated from linear (L)-measures – 8.42 Cartesian product of semi-rings – 8.43 Product space and product measure originating from measure spaces; (L)-integral on a product space

8.5 Fubini's theorem for product spaces; connections with probability theory 221

8.51 Preliminaries on μ-measures – 8.52 The theorem of Fubini for abstract Lebesgue integrals – 8.53 Use of the notions of this chapter in Kolmogorov's construction of probability theory

PART TWO

FUNCTION SPACES AND ORTHOGONAL EXPANSIONS

CHAPTER 9: FOUNDATIONS OF FUNCTIONAL ANALYSIS. THE SPACES l^2 AND L^2

9.1 Orthogonal series and functional analysis 229

9.11 The notions of orthogonal system and series – 9.12 Geometrical background and role of Lebesgue's theory – 9.13 Bearing on the formation of functional analysis

9.2 Vectorial interpretation of the Euclidean space E_ν 231

9.21 Vectorial basic operations, norm in the space E_ν – 9.22 Angle of inclination, orthogonality and projection – 9.23 Orthogonal system and its completeness – 9.24 Use of an orthogonal system as a base

9.3 Transfer of fundamental ideas to the Hilbert space l^2 235

9.31 Definition of Hilbert's sequence space; vectorial basic operations, norm and notions connected with orthogonality – 9.32 Concept of distance, triangle inequality, theory of point sets in the space l^2 – 9.33 The role of l^2-convergence in connection with the "decomposition problem"

9.4 Further generalization: the function space L^2 as a vector field 238

9.41 The Hilbert space of square integrable functions – 9.42 Vectorial basic operations, norm, fundamental inequalities – 9.43 Notions issuing from norm and orthogonality – 9.44 Example of an incomplete orthonormal system of functions

9.5 Introduction of the notions of distance and convergence in space L^2 ... 243

9.51 Distance and L^2-convergence – 9.52 Comparison of the L^2-convergence with uniform and pointwise convergence – 9.53 The Riesz–Fischer convergence criterion – 9.54 Strong and weak convergence – 9.55 The space L^2 as a set of abstract points

9.6 The problem of decomposition and the concept of general Fourier series 250

9.61 Decomposition of an element of the space L^2 in terms of a given orthogonal system – 9.62 The formal determination of coefficients – 9.63 General

Fourier series. – 9.64 Parallelism between fundamental notions of the theory of orthogonal series and their geometric analogues

CHAPTER 10: SPECIAL ORTHOGONAL SYSTEMS AND SERIES

10.1 The trigonometric base system 255

10.11 Orthogonality of the trigonometric base system – 10.12 Lagrange's lemma – 10.13 Completeness of the trigonometric system, the cosine and sine system

10.2 The ordinary (trigonometric) Fourier series and its complex form 259

10.21 Ordinary Fourier coefficients and Fourier series – 10.22 Complex form of the ordinary Fourier series – 10.23 Connection with complex power series and Laurent series

10.3 Legendre polynomials 262

10.31 Existence of a sequence of polynomials orthogonal in a finite interval – 10.32 Representation of Legendre polynomials by means of a Rodrigues formula – 10.33 Some important properties of the Legendre polynomials – 10.34 Generating function – 10.35 Recurrence formula and differential equation – 10.36 Norm and minimum property, Legendre series

10.4 Properties of polynomial systems orthogonal with respect to weight functions 270

10.41 Definition of the weighted orthogonality of polynomial sequences – 10.42 Recursive representation of orthogonal polynomials belonging to a given interval and to a prescribed weight function – 10.43 Basic features, minimum property – 10.44 Some pertinent abbreviations and notations, the class $L^2_{\rho(x)}(a, b)$

10.5 Jacobi, Laguerre and Hermite polynomials 276

10.51 Tchebyshev polynomials of the first and second kind, Tchebyshev's approximation – 10.52 Generalization: Gegenbauer (ultraspherical) and Jacobi (hypergeometic) polynomials – 10.53 Infinite basic intervals: Laguerre and Hermite polynomials – 10.54 Outline of the classical systems of orthogonal polynomials – 10.55 Generating functions and norms – 10.56 Fourier series relating to the classical orthogonal polynomial systems

10.6 Orthogonal systems arising from differential equations of Sturm–Liouville type 282

10.61 Main types of Sturm–Liouville boundary value problems – 10.62 Orthogonality of eigenfunctions – 10.63 Example: discussion of the differential equation of harmonic oscillation – 10.64 The orthogonal system formed from Bessel functions of the first kind, Fourier–Bessel series – 10.65 Main properties of orthogonal systems of Sturm–Liouville type, Sturm–Liouville series

10.7 The Gram–Schmidt orthogonalization procedure 287

10.71 Orthogonalization in the space $L^2(a, b)$ – 10.72 Expansion of an "arbitrary" function in terms of a system of functions prescribed "optionally"

10.8 Applications: systems of Haar's type, Rademacher and Walsh functions 290

10.81 Closure of an orthonormal system; complete orthogonal polynomial sequences obtained by orthogonalization – 10.82 Generating the Haar system from characteristic functions belonging to the dyadic partition of $[0, 1]$ – 10.83 Completeness of Haar's system – 10.84 The related $\{\mu^{(r)}_{k_1 \cdots k_\nu}\}$ family of orthogonal functions – 10.85 Rademacher's orthonormal system and its importance in

probability theory – 10.86 Extension of the Rademacher system due to Walsh, generalized Walsh systems and series

CHAPTER 11: PROPERTIES OF ARBITRARY ORTHOGONAL SYSTEMS AND SERIES IN HILBERT'S FUNCTION SPACE

11.1 Minimum property of the partial sums of a Fourier series 299

11.11 Preliminary comments – 11.12 Fundamental identity for partial sums of orthogonal series – 11.13 Minimum property of the partial sums of general Fourier series and Bessel's identity – 11.14 Idea of approximation with respect to square integral (Bessel's approximation)

11.2 Bessel's inequality and the Parseval–Hurwitz formula 301

11.21 The Bessel inequality and its geometrical interpretation – 11.22 The Parseval–Hurwitz formula and its extended form – 11.23 Historical remarks

11.3 L^2-convergence of general Fourier series 303

11.31 The general expansion theorem – 11.32 Consequences

11.4 The Riesz–Fischer theorem: isomorphy of Hilbert's function space and sequence space 304

11.41 Characterization of the elements of L^2 by a countable number of data – 11.42 The fundamental theorem of F. Riesz and E. Fischer – 11.43 The isomorphic connection of L^2 and l^2

11.5 Convergence almost everywhere of arbitrary orthogonal series 307

11.51 The need to use the classical concept of convergence – 11.52 The principal problems of convergence theory of orthogonal series – 11.53 The Rademacher–Menshov theorem

11.6 The convergence and summation problem of general Fourier series 310

11.61 The integral form of a partial sum of the general Fourier series and its kernel function – 11.62 The notions of Lebesgue function and singular integral – 11.63 Pertinent observations

CHAPTER 12: CONVERGENCE OF SPECIAL FOURIER SERIES

12.1 Elementary convergence theorems on ordinary Fourier series; the Riemann–Lebesgue lemma 313

12.11 Elementary remarks on trigonometric Fourier series – 12.12 The order of magnitude of Fourier constants in case of a k-times differentiable function; a simple expansion theorem – 12.13 Sharpenings; case of a function of bounded variation – 12.14 The Riemann–Lebesgue lemma – 12.15 Conclusions and additions

12.2 Dirichlet's formula and Riemann's localization theorem 319

12.21 Dirichlet's integral – 12.22 The localization theorem of Riemann – 12.23 Riemann's condition for the Fourier series representation of a function

12.3 Convergence tests of Dini, Dirichlet, Jordan and Lipschitz 323

12.31 Dini's and Jordan's condition – 12.32 Regular points and Lipschitz conditions – 12.33 Dirichlet's criterion – 12.34 Other local or global tests of convergence

12.4 Example of a continuous function with ordinary Fourier series divergent at some point 329

12.41 Representation by Fourier series and continuity – 12.42 Uniform continuity of partial sums of the series $\sum(\sin nx)/n$ – 12.43 The Weierstrass example – 12.44 Du Bois-Reymond singularity and Schwarz's example – 12.45 On points of divergence of the ordinary Fourier series of a continuous function

12.5 The problem of uniqueness; term-by-term integrability of ordinary Fourier series 335

12.51 Connection between sum and coefficients of a convergent trigonometric series – 12.51 Outline of some pertinent results, set of uniqueness – 12.53 Term-by-term integration of ordinary Fourier series – 12.54 Theorem of term-by-term differentiation

12.6 The conjugate series 339

12.61 The notion of conjugate series and integral form of its partial sums – 12.62 Convergence criterion of Pringsheim – 12.63 Consequences, theorem of F. Lukács

12.7 Ordinary Fourier series of some special functions and harmonic analysis 342

12.71 The function $\{x\}$ – 12.72 Bernoulli polynomials and their applications – 12.73 The Hurwitz zeta function and Hurwitz's formula – 12.74 Development of $\cos(vx)$ into cosine series and expansion of cotangent in terms of partial fractions – 12.75 Fourier series of $|\sin x|$ – 12.76 Lerch's zeta function and its functional equation – 12.77 Remarks on harmonic analysis; generation of Bessel functions by complex Fourier series

12.8 Series of Haar's type; equiconvergence theorems for Sturm–Liouville series 355

12.81 Representation of continuous functions by Haar's series – 12.82 Proof of Haar's expansion theorem on the basis of the connection between Haar's system and some system of characteristic functions – 12.83 Convergence theorem for Fourier series of type $\mu^{(r)}_{k_1 \ldots k_\nu}$ – 12.84 Haar's equiconvergence theorem and its analogues for series of Sturm–Liouville type

CHAPTER 13: SUMMATION OF ORDINARY FOURIER SERIES

13.1 Linear summation methods and problems of Tauber's type 361

13.11 Introductory observations – 13.12 Foundations of the general theory of summation processes – 13.13 Classical summation methods: Cesàro's, Hölder's and Abel–Poisson summation – 13.14 Historical comments on the development of linear methods of summation – 13.15 Tauberian problems, theorem of Hardy and Landau

13.2 $(C, 1)$-summation of ordinary Fourier series; Fejér's fundamental and approximation theorem 369

13.21 Closed form for the arithmetic means of the partial sums of ordinary Fourier series: Fejér's integral – 13.22 The summation theorems of L. Fejér .

13.3 Some consequences of Fejér's theorem 372

13.31 Applications relating to the limit of partial sums and to the codomain of their arithmetic means for ordinary Fourier series – 13.32 The Gibbs phenomenon – 13.33 Concluding from the $(C, 1)$-summability of Fourier series to its convergence and to the approximation theorem of Weierstrass

13.4 Lebesgue's summation theorem and other generalizations 376
13.41 Local absolute and ordinary integral mean – 13.42 The summation theorem of Lebesgue and its proof found by Fejér – 13.43 M. Riesz's extension for the Lebesgue theorem and "strong summation" of Fourier series in the sense of Hardy and Littlewood

13.5 Summation of ordinary Fourier series by the Abel–Poisson method 380
13.51 Poisson's integral formula and its simplest consequences – 13.52 A sufficient condition for (A)-summability of Fourier series – 13.53 Connection with Cesàro's summation and with Dirichlet's problem in the plane

13.6 Application of the recent (D)-method 384
13.61 On certain insufficiencies of the classical summation processes – 13.62 Definition of the (D)-summation and its connection with zeta-functions – 13.63 The pertinent singular integral and its kernel function – 13.64 (D)-summability of ordinary Fourier series – 13.65 Remarks on effectiveness of the (D)-method; related summation processes and generalized integro-differentiation

CHAPTER 14: FOURIER INTEGRALS

14.1 Foundations of the theory of integral transforms of Fourier and related type 395
14.11 Trigonometric integrals and Fourier transforms, connection with the Laplace transformation – 14.12 Formal background of Fourier integrals as well as of Fourier's integral theorem and inversion formulae

14.2 Fourier transformation in Hilbert's function space 398
14.21 Complex L-space and L^2-space – 14.22 The Parseval formula for Fourier integrals – 14.23 Plancherel's theorem and Paley–Wiener theorem

14.3 Properties of absolutely convergent Fourier transforms 408
14.31 The complex Fourier transforms as linear operators – 14.32 The notion of convolution and its simplest properties – 14.33 Proof of the main theorem on convolutions

14.4 Fourier's integral theorem on classical convergence conditions 414
14.41 Transfer of the convergence theory of ordinary Fourier series – 14.42 Cesàro summation of integrals – 14.43 Analogue of the Fejér–Lebesgue summation theorem

14.5 Non-absolutely convergent Fourier integrals 423
14.51 The fundamental problem – 14.52 A general equiconvergence theorem – 14.53 Invertibility of non-absolutely convergent Fourier transforms – 14.54 Complementary results; generalization of Poisson's summation formula

14.6 Fourier–Stieltjes transforms 432
14.61 Fourier–Stieltjes transformation as a common extension of the Fourier transformation and ordinary Fourier series – 14.62 Probabilistic aspects, relation between distribution and characteristic function of a random variable

CHAPTER 15: FURTHER ABSTRACT SPACES, APPLICATIONS

15.1 Complex L^p-spaces of F. Riesz 437

15.11 Grounds for a common extension of $L(a,b)$ and $L^2(a,b)$ – 15.12 Main points of the theory of complex L^p-spaces – 15.13 Riesz's theorems on bounded linear functionals – 15.14 The sequence spaces l^p of Riesz

15.2 Linearity, metrics and completeness; Banach's fixed point theorem 441

15.21 The most important structural properties of abstract spaces – 15.22 General concept of linear (vector) space as well as of metric and complete space – 15.23 The Banach fixed point theorem for complete metric spaces

15.3 Abstract Hilbert space and Neumann's foundation of quantum mechanics 444

15.31 The abstract Hilbert space H as a common generalization of L^2 and l^2 – 15.32 The Fréchet–Riesz theorem on linear operators in H and spectral decomposition theorem – 15.33 Linear algebraical interpretation of the theory – 15.34 Applications: equivalence of matrix and wave mechanics, the foundations of quantum mechanics by J. Neumann – 15.35 On the use of Dirac's impulse δ and related operators

15.4 Banach space; concept of general topological space 450

15.41 The Banach space B as a further extension of H and the spaces L^p; theorem of J. Neumann and P. Jordan – 15.42 Linear operators in B, dual Banach spaces – 15.43 Operator pendant of the geometric series (Neumann's series) – 15.44 General topological space and problem of metrization

15.5 Extended applications concerning ordinary differential equations 454

15.51 Introductory comments – 15.52 General form of self-adjoint eigenvalue problems – 15.53 Extended orthogonality relations for eigenfunctions – 15.54 Transformation of self-adjoint eigenvalue problems into integral equations via Green functions

15.6 Partial differential equations and orthogonal series 457

15.61 Solution of the problem of homogeneous vibrating string by Fourier's method – 15.62 Case of inhomogeneous string, use of Sturm–Liouville series – 15.63 Discussion of Dirichlet's problem in the plane by harmonic series of functions – 15.64 The Laplace (potential) equation in space; spherical harmonics and Laplace series – 15.65 Fourier's solution of the problem of linear heat conduction and of similar problems – 15.66 Membrane vibration and Fourier–Bessel series; boundary problems of further partial differential equations of second order in physics

15.7 Solving integral equations by operator methods 465

15.71 Linear integral equation of Fredholm type – 15.72 C.G. Neumann's existence and expansion theorem; Fredholm's alternative – 15.73 Case of symmetric kernel function, the Hilbert–Schmidt expansion theorem

BIBLIOGRAPHY 471

INDEX 483