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### Frequently used notation

- $\mathcal{A}$  — is the mod  $p$  Steenrod algebra  
 $\mathcal{A}_*$  — is the dual of the mod  $p$  Steenrod algebra  
 $\mathcal{Alg}$  — is the category of unital, commutative,  $\mathbb{N}$ -graded  $\mathbb{F}_p$ -algebras  
 $\mathcal{Alg}_a$  — is the category of augmented objects in  $\mathcal{Alg}$   
 $\mathcal{E}$  — is the category of  $\mathbb{F}_p$ -vector spaces  
 $\mathcal{E}_{\text{gr}}$  — is the category of graded  $\mathbb{F}_p$ -vector spaces  
 $H^*X$  — is the mod  $p$  cohomology of the space  $X$   
 $\tilde{H}^*X$  — is the reduced mod  $p$  cohomology of the space  $X$   
 $H^*V \cong H^*BV$  — is the mod  $p$  cohomology of  $V$   
 $\tilde{H}^*V$  — is the reduced mod  $p$  cohomology of  $V$   
 $\mathcal{K}$  — is the category of unstable  $\mathcal{A}$ -algebras  
 $\mathcal{K}_a$  — is the category of augmented objects in  $\mathcal{K}$   
 $M^{\oplus a}$ ,  $a \in \mathbb{N}$  (or more generally a cardinal) — denotes the direct sum  
of  $a$  copies of the object  $M$  of  $\mathcal{E}$ , or of  $\mathcal{E}_{\text{gr}}$ , or of  $\mathcal{U}$  etc.  
 $\mathcal{O}$  — denotes always “a” forgetful functor  
 $\mathcal{U}$  — is the category of unstable  $\mathcal{A}$ -modules  
 $\mathcal{U}'$  — is the category of evenly graded unstable  $\mathcal{A}$ -modules  
 $V, W, \dots$  — denote finite dimensional  $\mathbb{F}_p$ -vector spaces  
 $\alpha(n)$ ,  $n \in \mathbb{N}$  — is the sum of the coefficients in the  $p$ -adic expansion  
of the integer  $n$   
 $\twoheadrightarrow$  — always denotes an epimorphism  
 $\hookrightarrow$  — always denotes a monomorphism