

## CONTENTS

Foreword	
<i>by S. S. Chern</i>	v
<i>by C. C. Hsiung</i>	vii
Chapter I. Basic Properties of Convex Sets	1
1.1 Basic Concepts	1
1.1.1 Convex sets and convex curves	1
1.1.2 Support lines and their existence	1
1.2 Support Functions and Width Functions	2
1.2.1 Generalized normal equations of lines	2
1.2.2 Support function and width function of a convex set	3
1.2.3 Convex curve as envelope of a family of lines	4
1.2.4 An elementary proof of the formula of perimeter	5
1.3 Some Special Convex Sets	7
1.3.1 Convex sets of constant widths	7
1.3.2 Parallel convex sets	8
1.4 Mixed Areas of Minkowski	8
1.4.1 Mixed convex sets	8
1.4.2 Mixed areas of Minkowski	8
1.5 Surface Area of the Unit Sphere and Volume of the Unit Ball	10
Chapter II. Measure for Sets of Geometric Elements	11
2.1 Measure for Sets of Points	11
2.1.1 Measure for sets of points	11
2.1.2 Remarks	12
2.1.3 An integral formula	13
2.2 Measure for Sets of Lines	15
2.2.1 Measure for sets of lines	15
2.2.2 Two corollaries	16
2.2.3 Other forms of density for sets of lines	17
2.2.4 Proof of isoperimetric inequality	18

2.3	Pairs of Points and Lines	19
2.3.1	Density of pairs of points	19
2.3.2	Integral for the power of chords of a convex set	20
2.3.3	Remarks on integrals for power of chords of a convex set	21
2.3.4	Inequalities for the integrals of the power of chord of a convex set	22
2.3.5	Density for pairs of lines	22
2.3.6	Crofton's formula	22
2.4	Division of the Plane by Random Lines	23
2.4.1	Division of the convex set by random lines	23
2.4.2	Division of plane by random lines	26
2.4.3	Notes on random division	28
2.5	Sets of Strips in the Plane	28
2.5.1	Density for sets of strips	28
2.5.2	Generalized Buffon's needle problem	29
2.5.3	Further generalizations	29
Chapter III. Fundamental Formulas of Integral Geometry in the Plane		33
3.1	The Group of Motions in the Plane	33
3.1.1	The group of motions in the plane	33
3.1.2	Left and right translations	34
3.1.3	The differential forms on $\mathcal{M}$	35
3.2	The Kinematic Density	36
3.2.1	Left and right invariant 1-forms	36
3.2.2	The kinematic density	37
3.2.3	Geometrical meaning of the kinematic measure	38
3.2.4	Other expressions for the kinematic measure	39
3.3	Poincaré's Formula	40
3.3.1	A new expression for the kinematic density	40
3.3.2	Poincaré's Formula	42
3.4	The Fundamental Kinematic Formula of Blaschke	43
3.4.1	Total curvature of a closed curve and of a plane domain	43
3.4.2	Fundamental kinematic formula of Blaschke	44
3.4.3	Some immediate consequences of Blaschke's formula	46
Chapter IV. Applications of Integral Geometry in the Plane		48
4.1	The Isoperimetric Inequality	48
4.1.1	The integral geometric proof of the isoperimetric inequality	48
4.1.2	Stronger isoperimetric inequalities	49
4.1.3	An upper limit for isoperimetric deficit	51
4.2	Conditions for One Domain to be Able to Contain Another	52
4.2.1	Sufficient conditions for one domain to contain another	53

4.2.2	Hadwiger's conditions	54
4.2.3	Some consequences	55
4.2.4	Pseudodiameter of a domain	55
4.3	Kinematic Measure of a Segment of Fixed Length within a Convex Domain	56
4.3.1	Problem	56
4.3.2	A formula for kinematic measure of a segment within a convex domain	56
4.3.3	Generalized support function and the restricted chord function	58
4.3.4	A formula for $m(l)$ expressed by the generalized support function	58
4.3.5	The measure $m(l)$ for a rectangle	63
4.4	Applications of $m(l)$ to Geometric Probability	63
4.4.1	Laplace extension of Buffon problem	63
4.4.2	Applications of $m(l)$ to generalized Buffon problem	64
4.4.3	Formulas of $m(l)$ for equilateral triangle and regular hexagon	66
4.5	A Problem Related to the Statistical Estimating of $\pi$	67
4.5.1	Equidistant parallel lines	67
4.5.2	Grid of rectangles, independence	67
4.5.3	Efficiency analysis	68
4.6	Random Convex Set in a Lattice of Parallelograms	70
4.6.1	Width functions of convex sets	70
4.6.2	Distribution of the number of intersections	72
4.6.3	Hitting probabilities	76
4.6.4	Independence	77
Chapter V. Foundations of Integral Geometry in Homogeneous Spaces		80
5.1	Differentiable Manifolds	80
5.1.1	Topological space	80
5.1.2	Topological manifolds and differentiable manifolds	81
5.1.3	Differentiable functions and mappings	82
5.2	Vector Fields on a Manifold	82
5.2.1	Tangent spaces and vector fields	82
5.2.2	The differential of a mapping between manifolds	83
5.2.3	Local expressions of vector fields	83
5.3	Differential Forms and Exterior Differentiation	84
5.3.1	Covector fields	84
5.3.2	Tensor fields	85
5.3.3	The exterior algebra on a manifold	86
5.3.4	Exterior differentiation	89
5.3.5	Expression of exterior differential by the usual differential	90

5.4	Integral Manifolds and Pfaffian Systems	91
5.4.1	Integral manifolds	91
5.4.2	Pfaffian system	92
5.5	Lie Groups and Kinematic Density of a Lie Group	95
5.5.1	Lie group	95
5.5.2	Left and right translations	95
5.5.3	Left invariant differential forms	97
5.5.4	Structure equations and structure constants for a Lie group	99
5.5.5	Kinematic density for Lie group	105
5.6	Density and Measure in Homogeneous Space	110
5.6.1	Actions of a Lie group on a manifold, homogeneous space	110
5.6.2	Conditions for the existence of invariant density on $G/H$	111
5.6.3	Weil's condition	113
5.6.4	Normal subgroups	115
5.6.5	Chern's conditions	115
5.6.6	Stable subgroups	116
5.7	A Brief Review of Integral Geometry in the Plane	117
Chapter VI. Integral Geometry in $\mathbb{R}^n$		121
6.1	The Group of Motions in $\mathbf{E}_n$	121
6.1.1	The group of motions and its structure equations	121
6.1.2	Invariant volume elements of the group of motions and its subgroups	123
6.2	The Density of $r$ -Planes in $\mathbf{E}_n$	125
6.2.1	The density of $r$ -planes	125
6.2.2	Density of $r$ -planes about a fixed $q$ -plane	126
6.2.3	The volume of the Grassmann manifold $G_{r,n-r}$	127
6.2.4	Another form of the density of $r$ -planes in $\mathbf{E}_n$	128
6.2.5	The density of the pairs of hyperplanes	129
6.2.6	The density of the pairs $(L_r, L_{i+1}^{(r)})$	130
6.2.7	A density formula for points and flats	131
6.2.8	The density of the pairs of non-intersecting flats	133
6.3	Convex Sets in $\mathbf{E}_n$	134
6.3.1	Convex sets and mixed volumes	134
6.3.2	Quermassintegrale	136
6.3.3	Cauchy's formula and Steiner's formula	140
6.3.4	The mean value of $W'_i(K'_{n-r})$	142
6.4	Integrals of Mean Curvature	143
6.4.1	Integrals of mean curvatures of hypersurfaces in $\mathbf{E}_n$	143
6.4.2	Relations between integrals of mean curvature and quermassintegrale	145
6.4.3	Some particular results	146

6.4.4	Integrals of mean curvature of a flattened convex body	148
6.5	Sets of $r$ -Planes that Intersect a Convex Set	149
6.5.1	Measures of the sets of $r$ -planes that intersect a convex set	149
6.5.2	The integral of $W_{i+1}^{(r)}(L_r \cap K)$ over the set $\{L_r : L_r \cap K \neq \emptyset\}$	150
6.5.3	Crofton and Hadwiger's formula	151
6.6	Chern's Formulas	152
6.6.1	A density formula	152
6.6.2	The integral of $\Delta^{r+q-n}$	153
6.6.3	Chern's formula	154
6.7	Santaló's Formula	156
6.7.1	A density formula	156
6.7.2	Santaló's formula	157
6.8	Integral of the Volume of the Intersection of Two Manifolds	159
6.8.1	A density formula	159
6.8.2	Another density formula	160
6.8.3	Integral of the volume $\sigma_{r+q-n}(M^r \cap M^q)$	161
6.9	Chern-Yen's Kinematic Fundamental Formula	162
6.9.1	An important density formula	162
6.9.2	Chern and Yen's kinematic fundamental formula	164
6.9.3	Kinematic fundamental formula for convex sets	170
6.9.4	Integral formulas for the integrals of mean curvature	171
Chapter VII. Applications of Integral Geometry		173
7.1	Introduction to Integral Geometry in $\mathbb{R}^3$	173
7.1.1	Group of the motions in $\mathbb{R}^3$	173
7.1.2	Densities for lines and planes in $\mathbb{R}^3$	175
7.1.3	Some fundamental formulas	177
7.1.4	Moving cylinders	179
7.2	Elements of Stereology	181
7.2.1	Objects of study in stereology	181
7.2.2	General discussion	181
7.2.3	Intersection with random planes	183
7.2.4	Spherical particles	186
7.2.5	Nearly spherical particles	187
7.2.6	Intersection with random lines	188
7.2.7	Estimation for the number of crystals	189
7.3	Sufficient Conditions for One Domain to Contain Another	190
7.3.1	A density formula	191
7.3.2	A sufficient condition for one convex body to contain another	192
7.3.3	Sufficient conditions for one domain to contain another in $\mathbb{R}^3$	195
7.3.4	Analogues of Hadwiger's theorem in $\mathbb{R}^n$ ( $n \geq 4$ )	200

7.4 Kinematic Measure of a Segment of Fixed Length Within a Convex Body	201
7.4.1 A general formula for the kinematic measure of a segment of fixed length within a convex body	201
7.4.2 Transformation of formula	203
7.4.3 Formulas of $m(l)$ for a cylinder	206
7.4.4 Kinematic measure $m(l)$ for a right parallelepiped in $\mathbb{R}^3$ and Buffon problem	208
7.4.5 Kinematic measure $m(l)$ for a right parallelepiped in $\mathbb{R}^n$ and Buffon problem	212
7.5 Unified Inequalities Relating to Integrals for the Power of Chords	213
7.5.1 Inequalities relating to integrals for the power of chords in $\mathbb{R}^3$	214
7.5.2 Applications to geometric probability	217
7.5.3 Inequalities of integrals for the power of chords in $\mathbb{R}^n$	218
7.5.4 Integral geometric inequalities for moments	223
7.5.5 Pairs of non-intersecting random flats meeting two convex bodies	227
7.6 Inequalities Characterizing Simplices	228
7.6.1 Lemmas	228
7.6.2 Inequalities characterizing simplices	230
Index	234