

Contents

Acknowledgements	xi
Basic notation	xii
Introduction	1
1. General measure theory	7
Some basic notation	7
Measures	8
Integrals	13
Image measures	15
Weak convergence	18
Approximate identities	19
Exercises	22
2. Covering and differentiation	23
A $5r$ -covering theorem	23
Vitali's covering theorem for the Lebesgue measure	26
Besicovitch's covering theorem	28
Vitali's covering theorem for Radon measures	34
Differentiation of measures	35
Hardy–Littlewood maximal function	40
Measures in infinite dimensional spaces	42
Exercises	43
3. Invariant measures	44
Haar measure	44
Uniformly distributed measures	45
The orthogonal group	46
The Grassmannian of m -planes	48
The isometry group	52
The affine subspaces	53
Exercises	53
4. Hausdorff measures and dimension	54
Carathéodory's construction	54
Hausdorff measures	55
Hausdorff dimension	58
Generalized Hausdorff measures	59
Cantor sets	60
Self-similar and related sets	65
Limit sets of Möbius groups	69

Dynamical systems and Julia sets	71
Harmonic measure	72
Exercises	73
5. Other measures and dimensions	75
Spherical measures	75
Net measures	76
Minkowski dimensions	76
Packing dimensions and measures	81
Integralgeometric measures	86
Exercises	88
6. Density theorems for Hausdorff and packing measures	89
Density estimates for Hausdorff measures	89
A density theorem for spherical measures	92
Densities of Radon measures	94
Density theorems for packing measures	95
Remarks related to densities	98
Exercises	99
7. Lipschitz maps	100
Extension of Lipschitz maps	100
Differentiability of Lipschitz maps	100
A Sard-type theorem	103
Hausdorff measures of level sets	104
The lower density of Lipschitz images	105
Remarks on Lipschitz maps	106
Exercises	107
8. Energies, capacities and subsets of finite measure	109
Energies	109
Capacities and Hausdorff measures	110
Frostman's lemma in \mathbf{R}^n	112
Dimensions of product sets	115
Weighted Hausdorff measures	117
Frostman's lemma in compact metric spaces	120
Existence of subsets with finite Hausdorff measure	121
Exercises	124
9. Orthogonal projections	126
Lipschitz maps and capacities	126
Orthogonal projections, capacities and Hausdorff dimension	127
Self-similar sets with overlap	134
Brownian motion	136
Exercises	138
10. Intersections with planes	139
Slicing measures with planes	139

Plane sections, capacities and Hausdorff measures	142
Exercises	145
11. Local structure of s-dimensional sets and measures	146
Distribution of measures with finite energy	146
Conical densities	152
Porosity and Hausdorff dimension	156
Exercises	158
12. The Fourier transform and its applications	159
Basic formulas	159
The Fourier transform and energies	162
Distance sets	165
Borel subrings of \mathbf{R}	166
Fourier dimension and Salem sets	168
Exercises	169
13. Intersections of general sets	171
Intersection measures and energies	171
Hausdorff dimension and capacities of intersections	177
Examples and remarks	180
Exercises	182
14. Tangent measures and densities	184
Definitions and examples	184
Preliminary results on tangent measures	186
Densities and tangent measures	189
s -uniform measures	191
Marstrand's theorem	192
A metric on measures	194
Tangent measures to tangent measures are tangent measures ...	196
Proof of Theorem 11.11	198
Remarks	200
Exercises	200
15. Rectifiable sets and approximate tangent planes	202
Two examples	202
m -rectifiable sets	203
Linear approximation properties	205
Rectifiability and measures in cones	208
Approximate tangent planes	212
Remarks on rectifiability	214
Uniform rectifiability	215
Exercises	218
16. Rectifiability, weak linear approximation and tangent measures	220
A lemma on projections of purely unrectifiable sets	220

Weak linear approximation, densities and projections	222
Rectifiability and tangent measures	228
Exercises	230
17. Rectifiability and densities	231
Structure of m -uniform measures	231
Rectifiability and density one	240
Preiss's theorem	241
Rectifiability and packing measures	247
Remarks	247
Exercises	249
18. Rectifiability and orthogonal projections	250
Besicovitch–Federer projection theorem	250
Remarks on projections	258
Besicovitch sets	260
Exercises	264
19. Rectifiability and analytic capacity in the complex plane	265
Analytic capacity and removable sets	265
Analytic capacity, Riesz capacity and Hausdorff measures	267
Cauchy transforms of complex measures	269
Cauchy transforms and tangent measures	273
Analytic capacity and rectifiability	275
Various remarks	276
Exercises	279
20. Rectifiability and singular integrals	281
Basic singular integrals	281
Symmetric measures	283
Existence of principal values and tangent measures	284
Symmetric measures with density bounds	285
Existence of principal values implies rectifiability	288
L^p -boundedness and weak $(1, 1)$ inequalities	289
A duality method for weak $(1, 1)$	292
A smoothing of singular integral operators	295
Kolmogorov's inequality	298
Cotlar's inequality	299
A weak $(1, 1)$ inequality for complex measures	301
Rectifiability implies existence of principal values	301
Exercises	304
References	305
List of notation	334
Index of terminology	337