Contents

1 Preliminaries **1**

1.1 Notations . . . 1
1.2 The δ–decomposition of a set . . . 2
1.3 Notions related to Hausdorff Measures. Conditions $f.l.$ and $\sigma.f.l.$ . . . 2
1.4 Oscillations . . . 4
1.5 Borel sets F_σ, G_δ; Borel Functions; Analytic sets . . . 8
1.6 Densities; First Category sets . . . 9
1.7 The Baire Category Theorem; Romanovski's Lemma . . . 10
1.8 Vitali's Covering Theorem . . . 11
1.9 The generalized properties PG, $[PG]$, P_1; P_2G . . . 11
1.10 Extreme derivatives . . . 13
1.11 Approximate continuity and derivability . . . 14
1.12 Sharp derivatives $D^{\#}F$ . . . 15
1.13 Local systems; examples . . . 16
1.14 $\mathcal{S}$-open sets . . . 19
1.15 Semicontinuity; $\mathcal{S}$-semicontinuity . . . 21

2 Classes of functions **25**

2.1 Darboux conditions $\mathcal{D}$, $\mathcal{D}_-$, $\mathcal{D}_+$ . . . 25
2.2 Baire conditions $\mathcal{B}_1$, $\underline{\mathrm{B}}_1$, $\overline{\mathcal{B}}_1$ . . . 27
2.3 Conditions $\mathcal{C}_i$; $\mathcal{C}_i^*$; $[\mathcal{C}_iG]$; $[\mathcal{C}G]$ . . . 31
2.4 Conditions internal, internal*, $\mathcal{Z}_i$, uCM . . . 33
2.5 Conditions $\mathcal{D}_-\mathcal{B}_1$, $\mathcal{D}\mathcal{B}_1$ . . . 36
2.6 Conditions $\mathcal{B}_1$, $w\mathcal{B}_1$, $\mathcal{S}_2^-$ and $\mathcal{S}_2^+$-local systems . . . 39
2.7 Conditions VB, $\underline{\mathrm{VB}}$, VBG . . . 41
2.8 Conditions VB^*, $\underline{\mathrm{VB}}^*$, VB^*G . . . 44
2.9 Conditions monotone* and VB^* . . . 47
2.10 Conditions VB^*, VB^*G and $\mathcal{D}$, $\mathcal{D}_-$, $[CG]$, $[\mathcal{C}_iG]$, lower internal, internal 50
2.11 Conditions AC, ACG, . . . 51
2.12 Conditions AC^*, $\underline{\mathrm{AC}}^*$, AC^*G, $\underline{\mathrm{AC}}^*G$ . . . 54
2.13 Conditions L, $\underline{\mathrm{L}}$, LG, $\underline{\mathrm{L}}\mathrm{G}$ . . . 58
2.14 Summability and conditions VB and AC . . . 60
2.15 Differentiability and conditions VBG, VB^*G . . . 62
2.16 A fundamental lemma for monotonicity . . . 65
2.17 Krzyzewski's lemma and Foran's lemma . . . 70
2.18 Conditions (N), T_1, T_2, (S), $(+)$, $(-)$ . . . 71
2.19 Conditions wS, wN . . . 78

2.20 Condition $(\overline{N})$ . . . 78
2.21 Conditions N^{∞}, $N^{+\infty}$, $N^{-\infty}$ . . . 79
2.22 Conditions M^*, $\overline{M}^*$ . . . 82
2.23 Conditions (M), $\overline{M}$, N_g^{∞}, $N_g^{+\infty}$ . . . 84
2.24 Derivation bases . . . 87
2.25 Conditions $AC_{D\#}$, AC_{D^o}, AC_D . . . 88
2.26 Condition $Y_{D\#}$, Y_{D^o}, Y_D . . . 89
2.27 Characterizations of $AC^*G \cap \mathcal{C}$, $AC^*G \cap \mathcal{C}_i$, AC and $\underline{\mathrm{AC}}$ . . . 90
2.28 Conditions AC_n, AC_ω, AC_∞, $\mathcal{F}$ . . . 93
2.29 Conditions VB_n, VB_ω, VB_∞, $\mathcal{B}$ . . . 96
2.30 Variations V_n, V_ω, V_∞ and the Banach Indicatrix . . . 100
2.31 Conditions S_o, wS_o and AC_∞, VB_∞, (N) . . . 103
2.32 Conditions L_n, L_ω, L_∞, $\mathcal{L}$ . . . 104
2.33 Conditions ΛZ, $\Lambda\overline{Z}$, $f.l.$, $\sigma.f.l.$ . . . 108
2.34 Conditions SAC_n, SAC_ω, SAC_∞, $S\mathcal{F}$ . . . 111
2.35 Conditions SVB_n, SVB_ω, SVB_∞,; $S\mathcal{B}$ . . . 114
2.36 Conditions DW_n, DW_ω, DW_∞, DW^* . . . 116
2.37 Conditions E_n, E_ω, E_∞, $\mathcal{E}$ . . . 118
2.38 Conditions $\mathcal{SAC}$, $\mathcal{SACG}$, $\mathcal{SVB}$, $\mathcal{SVBG}$, $\mathcal{SY}$ . . . 121

3 Finite representations for continuous functions **127**
3.1 quasi-derivable $\subseteq AC^*$; DW^*G+AC^*; DW^*G and approximately quasi-derivable $\subseteq AC$; $DW_1G + AC$; DW_1G . . . 127
3.2 $\mathcal{C} \subseteq DW_1 + DW_1$ on a perfect nowhere dense set . . . 130
3.3 Wrinkled functions (W) and condition (M) . . . 131
3.4 $\mathcal{C}$ = quasi-derivable + quasi-derivable . . . 134
3.5 $\mathcal{C} = AC^*$; $DW_1G + AC^*$; $DW_1G + AC^*$; DW_1G . . . 135

4 Monotonicity **141**
4.1 Monotonicity and conditions $(-)$, $VB_\omega G$, $\mathcal{D}_\mathcal{B}_1$ . . . 141
4.2 Monotonicity and conditions $\underline{\mathrm{M}}$,uCM,$\underline{\mathrm{AC}}$,C_i, $\mathcal{C}_i^*$, $\mathcal{D}_i$ . . . 142
4.3 Monotonicity and conditions $N^{-\infty}$; N^{∞} . . . 145
4.4 Local monotonicity . . . 149
4.5 $\mathcal{S}$-derivatives and the Mean Value Theorem . . . 149
4.6 Relative monotonicity . . . 151
4.7 An application of Corollary 4.4.1 . . . 151
4.8 A general monotonicity theorem . . . 152
4.9 Monotonicity in terms of extreme derivatives . . . 158

5 Integrals **161**
5.1 Descriptive and Perron type definitions for the Lebesgue integral . . . 161
5.2 Ward type definitions for the Lebesgue integral . . . 166
5.3 Henstock variational definitions for the Lebesgue integral . . . 167
5.4 Riemann type definitions for the Lebesgue integral (The McShane integral) . . . 169
5.5 Theorems of Marcinkiewicz type for the Lebesgue integral . . . 172
5.6 Bounded Riemann$^{\#}$ sums and locally small Riemann$^{\#}$ sums . . . 173
5.7 Descriptive and Perron type definitions for the $\mathcal{D}^*$-integral . . . 174

5.8 An improvement of the Hake Theorem 180
5.9 An improvement of the Looman-Alexandroff Theorem. The equivalence of the $\mathcal{D}^*$-integral and the $(\mathcal{P}_{j,k})$-integral 184
5.10 Ward type definitions for the $\mathcal{D}^*$-integral 186
5.11 Henstock Variational definitions for the $\mathcal{D}^*$- integral 187
5.12 The Kurzweil-Henstock integral 188
5.13 Cauchy and Harnak extensions of the $\mathcal{D}^*$ - integral 189
5.14 A theorem of Marcinkiewicz type for the $\mathcal{D}^*$- integral 190
5.15 Bounded Riemann sums and locally small Riemann sums 192
5.16 Riemann type integrals and local systems 193
5.17 The $< LPG >$ and $< LDG >$ integrals 197
5.18 The chain rule for the derivative of a composite function 200
5.19 The chain rule for the approximate derivative of a composite function . 202
5.20 Change of variable formula for the Lebesgue integral 204
5.21 Change of variable formula for the Denjoy* integral 205
5.22 Change of variable formula for the $< LDG >$ integral 206
5.23 Integrals of Foran type 207
5.24 Integrals which extend both, Foran's integral and Iseki's integral 210

6 Examples **213**
6.1 The Cantor ternary set, a perfect nowhere dense set 213
6.2 The Cantor ternary function φ 214
6.3 A real bounded $\mathcal{S}_o^+$ closed set which is not of F_σ-type 214
6.4 An $\mathcal{S}_o^+$ lower semicontinuous function which is not $\overline{\mathcal{B}}_1$ 215
6.5 A function $F \in \mathcal{C}_i$, $F \notin \mathcal{C}_i^*$ 215
6.6 A function $F \in \mathcal{D}$, $F \in [\mathcal{C}_i^* G]$, $F \notin [\mathcal{C}G]$ 215
6.7 A function $F \in \mathcal{D}\mathcal{B}_1$, $F \notin [\mathcal{C}_i G]$ 216
6.8 A function $F \in uCM$; $F \notin \ell CM$ 216
6.9 A function concerning conditions $\mathcal{D}_+$, $\mathcal{D}_-$, CM, sCM, lower internal . 217
6.10 A function concerning conditions: $\mathcal{D}_-$, $\mathcal{D}$, internal, $\underline{\mathcal{B}}_1$, $\overline{\mathcal{B}}_1$, $\mathcal{B}_1$, $w\mathcal{B}_1$, $[VBG]$, $(-)$, T_1, T_2 (Bruckner) 217
6.11 A function concerning conditions: $\overline{\mathcal{B}}_1$, $\underline{\mathcal{B}}_1$, $\mathcal{D}_-$, $\mathcal{D}_+$, lower internal, internal, internal* (Dirichlet) 218
6.12 A function concerning conditions: $\mathcal{D}$, $\mathcal{D}_-$, $\mathcal{B}_1$, $\mathcal{C}_i$, $\mathcal{C}_i^*$, lower internal, internal*, VB, VB^*G, $N^{-\infty}$ 219
6.13 A function $F \in \mathcal{D}$, $F \in \underline{\mathcal{B}}_1 \setminus \overline{\mathcal{B}}_1$; $-F \in \overline{\mathcal{Z}}_i \setminus \mathcal{C}_i$, $-F \in \mathcal{D}_-\overline{\mathcal{B}}_1 \setminus \overline{\mathcal{Z}}_i$ 219
6.14 A function $F \in \underline{\mathcal{B}}_1 \setminus \overline{\mathcal{B}}_1$, $F \in$ lower internal, $F \notin \mathcal{D}_-$ 220
6.15 A function $F \in sCM$, $F \notin$ internal* 220
6.16 A function $F \in AC^*G \setminus AC$, $F \in \mathcal{C}_i^* \setminus \mathcal{D}$, $F \in sCM \setminus$ internal* 220
6.17 A function $F \in (D.C.)$, $F \in \mathcal{B}_1$, $F \notin m_2$, $F \notin \mathcal{D}$ 221
6.18 A function $F \in (+) \cap (-)$; $F \notin \mathcal{D}\mathcal{B}_1 T_2$ 221
6.19 A function $G \in \mathcal{D}$, $G \notin \underline{\mathcal{B}}_1$, $G \notin \overline{\mathcal{B}}_1$, $G'_{ap}(x)$ exists *n.e.*, $G'_{ap}(x) \geq 0$ *a.e.* (Preiss) 222
6.20 A function $H \in \mathcal{D}$, $H \notin \overline{\mathcal{B}}_1$, $H \notin \underline{\mathcal{B}}_1$, $H'_{ap}(x)$ exists on $(0,1)$ (Preiss) . . 222
6.21 A function $F \in \mathcal{D}\mathcal{B}_1$, $F(x) = 0$ *a.e.*, F is not identically zero (Croft) . . 223
6.22 A function $F \in \mathcal{D}$, $F \in [\mathcal{C}G]$, $F \in [VBG]$, $F \notin VB^*G$, $F \notin \mathcal{C}$ (Bruckner) 223
6.23 A function $F \in AC^*$, $F \notin VB^*$ 224

6.24 A function $F \in \mathcal{C}$, $F \in T_1$, $F \in VBG$, $F \notin VB^*G$ 224
6.25 A function $F \in [bAC^*G] \cap VB^*G \cap N^{-\infty}$ $F \notin$ lower internal 225
6.26 A function $F \in \mathcal{C} \cap (S) \cap LG$, $F \notin AC^*G$, $F'(x)$ does not exist on a set of positive measure, $F(x) + x \in LG$, $F(x) + x \notin T_1$ 225
6.27 A function $F \in (S) \cap \mathcal{C}$ such that the sum of F and any linear nonconstant function does not satisfy (N) (Mazurkiewicz) 226
6.28 A function $F \in (M)$, $F \notin T_2$ 227
6.29 Functions concerning conditions (M), AC, T_1, T_2, (S), (N), L, L_2G, VBG, $S\mathcal{F}$, quasi-derivable 229
6.30 A function $G \in N^{\infty}$, $F \notin (M)$, $F \notin (+)$ 237
6.31 Functions concerning conditions (S), (N), (M), T_1, T_2, ACG, AC_n, SAC_n, VB_2, VBG, SVB, $\mathcal{F}$, $S\mathcal{F}$ 238
6.32 A function $F \in$ lower semicontinuous, $F \in AC_2$, $F \notin \underline{AC}$ 244
6.33 A function $F_n \in L_{n+1}$ on a perfect set, $F_n \in VB_n$ on no portion of this set, $F_n \in L_{n+1}G$, $F_n \notin AC_nG$ on $[0,1]$ 245
6.34 Functions $F \in L_2G$, $G_s \in (N)$, $G_s' = F'$ *a.e.*, $G_s - F$ is not identically zero, $F \notin SACG$ 247
6.35 A function $F \in \underline{L}_2$, $F \notin T_2$, $F \notin \mathcal{B}$ 250
6.36 A function $F \in VB_2$ on C, $V_2(F;C) \leq 1$ 252
6.37 A function $F_p \in L_{2^p}$, $F_p \notin AC_{2^p-1}$, $F_p \in VB_2$ on C; $V_2(F_p;C) \leq 1$ . . . 252
6.38 A function $G \in VB_2$, $G \notin AC_n$ on C, $G \in \mathcal{F}$ on $[0,1]$ 254
6.39 A function $F_1 \in VB_2$ on C, $V_2(F_1;[0,x] \cap C) = \varphi(x)$ (G. Ene) 254
6.40 A function $F_q \in (N)$ on $[0,1]$, $F_q \notin VB_n$ on C, $F_q \in VB_\omega$ on C (G. Ene) 255
6.41 A function $G_1 \in L_2G$, $G_1 \in \mathcal{F}$, $G_1 \notin SVBG$, $G_1 \notin SACG$, $(G_1)'_{ap}$ does not exist on a set of positive measure 256
6.42 A function $F \in SACG$, $F \notin \mathcal{F}$, $F \notin ACG$ 258
6.43 A function $F \in DW_1$, $F \notin DW^*$ 261
6.44 A function $F \in AC^*; DW_1G$, $F \notin AC^*; DW^*G$ 261
6.45 Functions $F_1, F_2 \in \mathcal{C} \cap AC^*; DW^*G$, F_1, F_2 are derivable *a.e.*, $F_1' = F_2'$ *a.e.*, F_1 and F_2 do not differ by a constant 262
6.46 Functions $F_1, F_2 \in \mathcal{C} \cap AC^*; DW_1G$, F_1, F_2 are approximately derivable *a.e.*, $F_1 + F_2 \notin$ quasi-derivable 262
6.47 Functions $F_1, F_2 \in \mathcal{E} \cap \mathcal{B}$, $F_1, F_2 \notin \mathcal{F}$, $F_1 + F_2 \notin \mathcal{E}$ 264
6.48 A function $G_n \in E_{n+1}$, $G_n \notin E_n$, $G_n \in L_{n^2+2n+1}$, $G_n \notin VB_{n^2+2n}$ 265
6.49 Functions concerning conditions $\underline{L}$,$\underline{F}$,$\overline{\mathcal{F}}$, VB_2G, $\mathcal{B}$, E_1G 268
6.50 A function $F \in \mathcal{E} \cap VB_\omega G$, $F \notin \mathcal{B}$ 270
6.51 A function $F \in (N)$, $F \notin \Lambda Z$ (Foran) 271
6.52 A function $F \in AC \circ \Lambda Z$, $F \notin \Lambda Z$ (Foran) 272
6.53 A function $H \in AC + \Lambda Z$, $H \notin \Lambda Z$ (Foran) 272
6.54 A function $G \in AC \cdot \Lambda Z$, $G \notin \Lambda Z$ (Foran) 272
6.55 A function $F_1 \in AC$, $F_1 \notin L_n$, $F_1 \notin \mathcal{L}$ 273
6.56 Functions $F_1 \in AC_2G$, $F_2 \in \Lambda Z$, $F_1 + F_2 \notin (M)$, $F_1' = -F_2'$ *a.e.* 273
6.57 A function $F \in \Lambda Z$, $F \notin [\mathcal{E}]$ 274
6.58 Functions $F_1 \in (S)$, $F_1 \in AC \circ \sigma.f.l.$, $F_1 \notin \sigma.f.l.$, $F_2 \in L$, $F_1 + F_2 \notin T_2$ 276
6.59 Functions $G_1 \in \sigma.f.l.$, $G_2 \in AC$, $G_1 + G_2 \notin \sigma.f.l.$ 279
6.60 Functions $H_1 \in \sigma.f.l.$, $H_2 \in AC$, $H_1 \cdot H_2 \notin \sigma.f.l.$ 280

6.61 A function $F \in \sigma.f.l.$, $F \in T_1$, $F \notin \mathcal{B}$, F is nowhere approximately derivable, (Foran) 280
6.62 A function $G \in \sigma.f.l.$, $G \in T_1$, G is nowhere derivable, $G'_{ap}(x) = 0$ *a.e.*, $G \notin W$, $G \in W^*$ (Foran) 284
6.63 A function $F \in W$ on a perfect nowhere dense set of positive measure, with each level set perfect, F is nowhere approximately derivable 287
6.64 A function $G_1 \in DW_1 \cap \mathcal{C}$, G_1 is not approximately derivable *a.e.* on a set of positive measure 291
6.65 A function $F \in \mathcal{C}$, F is quasi-derivable, $F \notin AC \circ AC + AC$ 291
6.66 Examples concerning the chain rule for the approximate derivative of a composite function 291

Bibliography **293**

Index **305**