

# Contents

Preface	xi
Preface to the First Edition	xiii
PART ONE INTRODUCTION, HISTORICAL BACKGROUND, AND NOTATIONS	
Chapter 1 Introduction and Historical Background	3
1. Number Theory—The Queen of Mathematics	3
2. A Problem	4
3. Something About the Contents of This Book	6
4. Number Theory and Other Branches of Mathematics	6
5. The Distribution of Prime Numbers	7
6. Fermat's "Last Theorem"	9
7. The Theory of Partitions	10
8. Elementary Number Theory	11
Chapter 2 Introductory Remarks and Notations	13
PART TWO ELEMENTARY NUMBER THEORY	
Chapter 3 Divisibility	19
1. Generalities and Fundamental Theorem	19
2. Discussion of Two Objections	25
3. An Example of Hilbert	26
4. Two Further Theorems	28
5. Some Results Concerning the Distribution of Prime Numbers	28
Chapter 4 Congruences	35
1. Congruences as Equivalence Relations—General Properties	35
2. Operations with Residue Classes	39
3. Theorems of Fermat, Euler, and Wilson	43
4. Linear Congruences	45
5. The Chinese Remainder Theorem	48
6. On Primitive Roots	50
7. *Congruences of Higher Degrees	55
Chapter 5 Quadratic Residues	67
1. Introduction	67
2. The Legendre Symbol and the Law of Quadratic Reciprocity	68
3. *The Jacobi and Kronecker Symbols	74
Chapter 6 Arithmetical Functions	81
1. Introduction	81
2. The Function $[x]$	81
3. The Function $y = ((x))$	83
4. The Euler Function $\phi(n)$	85
5. The Möbius Function $\mu(n)$	87
6. Liouville's Function	92

7. The Function $\delta_x(n)$	92
8. *Perfect Numbers	93
9. *Ramanujan Sums	94
10. Functions Related to Prime Numbers	97
11. Formulae that Yield Primes	101
12. **On the Sum Function $M(x)$	102
 Chapter 7 The Theory of Partitions	 109
1. Introduction	109
2. Definition and Notations	109
3. Survey of Methods	112
4. Generating Functions	112
5. Graphs	124
6. The Size of $\rho(n)$	127
7. Some Lemmas	130
8. Proof of the Theorem	132
 PART THREE TOPICS FROM ANALYTIC AND ALGEBRAIC NUMBER THEORY	
 Chapter 8 The Distribution of Primes and the Riemann Zeta Function	 143
1. The Distribution of Primes and the Sieve Method	143
2. From Tchebycheff to Landau	144
3. The Riemann Zeta Function	145
4. The Zeta Function and $\pi(x)$	148
5. Further Theory of the Zeta Function	149
6. The Functional Equation	153
7. The values of $\zeta(s)$ at Integral Arguments	155
8. The Zeta Function and Its Derivatives for $\sigma$ Close to 1	158
9. Comments on the Zeta Function	164
 Chapter 9 The Prime Number Theorem	 169
1. Introduction	169
2. Sketch of the Proof	170
3. Some Lemmas	173
4. Proof of the PNT	177
 Chapter 10 The Arithmetic of Number Fields	 187
1. Introduction	187
2. Fields and Rings of Algebraic Numbers	189
3. The Quadratic Field $Q(\sqrt{2})$	192
4. The Ring $I(\sqrt{2})$ of Integers of $K = Q(\sqrt{2})$ and its Units	195
5. Factorization in $K = Q(\sqrt{2})$	197
6. The Euclidean Algorithm in $Q(\sqrt{2})$	199
7. The Field $Q(\sqrt{-5})$	200
8. The Field $Q(\sqrt{5})$	204
9. The Gaussian Field $K = Q(\sqrt{-1})$	205
10. Quadratic Fields	207
11. Cyclotomic Fields	208
12. Other Algebraic Number Fields	211
13. Integer and Units in Algebraic Number Fields	213

Chapter 11	Ideal Theory	220
1.	Introduction	220
2.	Definition and Elementary Properties of Ideals	220
3.	Divisibility Properties of Ideals	222
4.	Uniqueness of Factorization of Ideals into Prime Ideals	224
5.	Ideal Classes and the Class Number	225
Chapter 12	Primes in Arithmetic Progressions	233
1.	Introduction and Dirichlet's Theorem	233
2.	Characters	235
3.	Dirichlet's $L$ -Functions	238
4.	Proof of Theorem 1	243
5.	Some Auxiliary Results	243
6.	End of the Proof of Theorem 1	247
7.	Dirichlet's Approach	248
8.	Concluding Remarks	249
Chapter 13	Diophantine Equations	255
1.	Introduction	255
2.	Hilbert's Tenth Problem	257
3.	Effective Methods	258
4.	The Genus of a Curve	259
5.	The Theorems and Conjectures of A. Weil	262
6.	The Hasse Principle	263
7.	Linear and Quadratic Diophantine Equations	264
8.	Representations by Sums of Two Squares	265
9.	Representations by Sums of Three Squares	267
10.	A Theorem of Legendre	269
11.	Proof of Legendre's Theorem	270
12.	Lagrange's Theorem and the Representation of Integers as Sums of Four Squares	273
13.	Representations of Integers as Sums of $r$ Squares	277
14.	The Mordell-Weil Theorem	277
15.	Bachet's Equation	281
16.	Final Remarks	285
Chapter 14	Fermat's Equation	289
1.	Introduction	289
2.	Solution of Fermat's Equation for $n = 2$	291
3.	Proof of the FC for $n = 4$	292
4.	Proof of the FC for $p = 3$	293
5.	Case I for $p = 5$	298
6.	Case I for an Arbitrary Prime $p$	300
7.	Regular Primes	304
8.	Proof of the FC for Regular Primes in Case I	308
9.	Proof of the FC for Regular Primes in Case II	310
10.	Some Final Remarks	315
	Answers to Selected Problems	321
	Subject Index	327
	Name Index	331
	Index of Functions	335