Contents

1 Sense, Denotation and Semantics 1
1.1 Sense and denotation in logic . . . 1
1.1.1 The algebraic tradition . . . 3
1.1.2 The syntactic tradition . . . 3
1.2 The two semantic traditions . . . 4
1.2.1 Tarski . . . 4
1.2.2 Heyting . . . 5

2 Natural Deduction 8
2.1 The calculus . . . 9
2.1.1 The rules . . . 10
2.2 Computational significance . . . 11
2.2.1 Interpretation of the rules . . . 11
2.2.2 Identification of deductions . . . 13

3 The Curry-Howard Isomorphism 14
3.1 Syntax . . . 15
3.1.1 Types . . . 15
3.1.2 Terms . . . 15
3.2 Denotational significance . . . 16
3.3 Operational significance . . . 17
3.4 Conversion . . . 18
3.5 Description of the isomorphism . . . 19
3.6 Relevance of the isomorphism . . . 20

4 The Normalisation Theorem 22
4.1 The Church-Rosser property . . . 22
4.2 The weak normalisation theorem . . . 24
4.3 Proof of the weak normalisation theorem . . . 24
4.3.1 Degree and substitution . . . 25
4.3.2 Degree and conversion . . . 25
4.3.3 Conversion of maximal degree . . . 26

4.3.4 Proof of the theorem 26
4.4 The strong normalisation theorem 26

5 Sequent Calculus 28
5.1 The calculus 29
5.1.1 Sequents 29
5.1.2 Structural rules 29
5.1.3 The intuitionistic case 30
5.1.4 The "identity" group 30
5.1.5 Logical rules 31
5.2 Some properties of the system without cut 33
5.2.1 The last rule 33
5.2.2 Subformula property 33
5.2.3 Asymmetrical interpretation 34
5.3 Sequent Calculus and Natural Deduction 35
5.4 Properties of the translation 39

6 Strong Normalisation Theorem 42
6.1 Reducibility 42
6.2 Properties of reducibility 43
6.2.1 Atomic types 43
6.2.2 Product type 44
6.2.3 Arrow type 44
6.3 Reducibility theorem 45
6.3.1 Pairing 45
6.3.2 Abstraction 45
6.3.3 The theorem 46

7 Gödel's system T 47
7.1 The calculus 48
7.1.1 Types 48
7.1.2 Terms 48
7.1.3 Intended meaning 48
7.1.4 Conversions 48
7.2 Normalisation theorem 49
7.3 Expressive power: examples 50
7.3.1 Booleans 50
7.3.2 Integers 50
7.4 Expressive power: results 52
7.4.1 Canonical forms 52
7.4.2 Representable functions 52

8 Coherence Spaces **54**
8.1 General ideas 54
8.2 Coherence Spaces 56
8.2.1 The web of a coherence space 56
8.2.2 Interpretation 57
8.3 Stable functions 58
8.3.1 Stable functions on a flat space 60
8.3.2 Parallel Or 60
8.4 Direct product of two coherence spaces 61
8.5 The Function-Space 62
8.5.1 The trace of a stable function 62
8.5.2 Representation of the function space 64
8.5.3 The Berry order 65
8.5.4 Partial functions 66

9 Denotational Semantics of T **67**
9.1 Simple typed calculus 67
9.1.1 Types 67
9.1.2 Terms 68
9.2 Properties of the interpretation 69
9.3 Gõdel's system 70
9.3.1 Booleans 70
9.3.2 Integers 70
9.3.3 Infinity and fixed point 72

10 Sums in Natural Deduction **73**
10.1 Defects of the system 74
10.2 Standard conversions 75
10.3 The need for extra conversions 76
10.3.1 Subformula Property 76
10.3.2 Extension to the full fragment 77
10.4 Commuting conversions 78
10.5 Properties of conversion 79
10.6 The associated functional calculus 80
10.6.1 Empty type (corresponding to $\perp$) 80
10.6.2 Sum type (corresponding to $\vee$) 81
10.6.3 Additional conversions 81

11 System F **82**
11.1 The calculus 82
11.2 Comments 83
11.3 Representation of simple types 84
11.3.1 Booleans 84

11.3.2 Product of types . . . 84
11.3.3 Empty type . . . 85
11.3.4 Sum type . . . 85
11.3.5 Existential type . . . 86
11.4 Representation of a free structure . . . 86
11.4.1 Free structure . . . 87
11.4.2 Representation of the constructors . . . 88
11.4.3 Induction . . . 88
11.5 Representation of inductive types . . . 89
11.5.1 Integers . . . 89
11.5.2 Lists . . . 91
11.5.3 Binary trees . . . 93
11.5.4 Trees of branching type *U* . . . 93
11.6 The Curry-Howard Isomorphism . . . 94

12 Coherence Semantics of the Sum **95**
12.1 Direct sum . . . 96
12.2 Lifted sum . . . 96
12.2.1 dI-domains . . . 98
12.3 Linearity . . . 99
12.3.1 Characterisation in terms of preservation . . . 99
12.3.2 Linear implication . . . 100
12.4 Linearisation . . . 101
12.5 Linearised sum . . . 103
12.6 Tensor product and units . . . 104

13 Cut Elimination (Hauptsatz) **105**
13.1 The key cases . . . 105
13.2 The principal lemma . . . 109
13.3 The Hauptsatz . . . 111
13.4 Resolution . . . 112

14 Strong Normalisation for F **114**
14.1 Idea of the proof . . . 115
14.1.1 Reducibility candidates . . . 115
14.1.2 Remarks . . . 115
14.1.3 Definitions . . . 116
14.2 Reducibility with parameters . . . 117
14.2.1 Substitution . . . 118
14.2.2 Universal abstraction . . . 118
14.2.3 Universal application . . . 118
14.3 Reducibility theorem . . . 119

15 Representation Theorem **120**

15.1 Representable functions 121

15.1.1 Numerals 121

15.1.2 Total recursive functions 122

15.1.3 Provably total functions 123

15.2 Proofs into programs 124

15.2.1 Formulation of $\mathbf{HA_2}$ 125

15.2.2 Translation of $\mathbf{HA_2}$ into **F** 126

15.2.3 Representation of provably total functions 127

15.2.4 Proof without undefined objects 129

A Semantics of System F **132**

A.1 Terms of universal type 132

A.1.1 Finite approximation 132

A.1.2 Saturated domains 133

A.1.3 Uniformity 134

A.2 Rigid Embeddings 134

A.2.1 Functoriality of arrow 136

A.3 Interpretation of Types 136

A.3.1 Tokens for universal types 137

A.3.2 Linear notation for tokens 138

A.3.3 The three simplest types 139

A.4 Interpretation of terms 141

A.4.1 Variable coherence spaces 141

A.4.2 Coherence of tokens 142

A.4.3 Interpretation of **F** 144

A.5 Examples 145

A.5.1 Of course 145

A.5.2 Natural Numbers 147

A.5.3 Linear numerals 148

A.6 Total domains 149

B What is Linear Logic? **150**

B.1 Classical logic is not constructive 150

B.2 Linear Sequent Calculus 152

B.3 Proof nets 155

B.4 Cut elimination 158

B.5 Proof nets and natural deduction 161

Bibliography **162**

Index and index of notation **166**