# Contents

*Foreword (by W. D. Brownawell)* ..... v

*Preface to the English edition* ..... ix

*Preface* ..... xi

*Notation* ..... xviii

*Introduction* ..... 1

§ 1. Approximation of algebraic numbers ..... 1
§ 2. The classical method of Hermite–Lindemann ..... 3
§ 3. Methods arising from the solution of Hilbert's Seventh Problem, and their subsequent development ..... 4
§ 4. Siegel's method and its further development ..... 7

*Chapter 1. Approximation of real and algebraic numbers* ..... 11

§ 1. Approximation of real numbers by algebraic numbers ..... 11
§ 2. Simultaneous approximation ..... 18
§ 3. Approximation of algebraic numbers by rational numbers ..... 23
§ 4. Approximation of algebraic numbers by algebraic numbers ..... 26
§ 5. Further refinements and generalizations of Liouville's Theorem ..... 35
Remarks ..... 38

*Chapter 2. Arithmetic properties of the values of the exponential function at algebraic points* ..... 41

§ 1. Transcendence of $e$ ..... 41
§ 2. Transcendence of $\pi$ ..... 46
§ 3. Transcendence of the values of the exponential function at algebraic points ..... 51
§ 4. Approximation of $e^z$ by rational functions ..... 58

§ 5. Linear approximating forms for $e^{\rho_1 z}, \ldots, e^{\rho_m z}$ ..... 63
§ 6. A set of linear approximating forms ..... 69
§ 7. Lindemann's Theorem ..... 70
§ 8. Linear approximating forms and the Newton interpolation series for the exponential function ..... 75
Remarks ..... 77

*Chapter 3. Transcendence and algebraic independence of the values of E-functions which are not connected by algebraic equations over the field of rational functions* ..... 79

§ 1. E-functions ..... 79
§ 2. The First Fundamental Theorem ..... 81
§ 3. Some properties of linear and fractional-linear forms ..... 85
§ 4. Properties of linear forms in functions which satisfy a system of homogeneous linear differential equations ..... 88
§ 5. Order of zero of a linear form at $z = 0$ ..... 93
§ 6. The determinant of a set of linear forms ..... 99
§ 7. Passing to linearly independent numerical linear forms ..... 101
§ 8. Auxiliary lemmas on solutions of systems of homogeneous linear equations ..... 102
§ 9. Functional linear approximating forms ..... 106
§ 10. Numerical linear approximating forms ..... 110
§ 11. Rank of the $m$-tuple $f_1(\xi), \ldots, f_m(\xi)$ ..... 115
§ 12. Proof of the First Fundamental Theorem ..... 118
§ 13. Consequences of the First Fundamental Theorem ..... 121
Remarks ..... 126

*Chapter 4. Transcendence and algebraic independence of the values of E-functions which are connected by algebraic equations over the field of rational functions* ..... 128

§ 1. Rank of the $m$-tuple $f_1(\xi), \ldots, f_m(\xi)$ ..... 128
§ 2. Some lemmas ..... 131
§ 3. Estimate for the dimension of a vector space spanned by monomials in elements of a field extension ..... 134
§ 4. The Third Fundamental Theorem ..... 139
§ 5. Transcendence of the values of E-functions connected by arbitrary algebraic equations over $\mathbf{C}(z)$ ..... 143
§ 6. Algebraic independence of the values of E-functions which are connected by arbitrary algebraic equations over $\mathbf{C}(z)$ ..... 147

§ 7. E-functions connected by special types of equations ..... 149
§ 8. E-functions connected by algebraic equations with constant coefficients ..... 151
§ 9. E-functions which are connected by a single algebraic equation over $\mathbf{C}(z)$ ..... 158
§ 10. Minimal equations ..... 163
§ 11. Dimension of the vector spaces spanned by monomials in the elements of a field extension ..... 168
§ 12. Algebraic independence of the values of IE-functions ..... 171
§ 13. Algebraic independence of the values of KE-functions ..... 174
Remarks ..... 177

*Chapter 5. Transcendence and algebraic independence of the values of E-functions which satisfy first order linear differential equations* ..... 179

§ 1. Hypergeometric E-functions ..... 179
§ 2. The simplest hypergeometric E-functions ..... 185
§ 3. Sets of solutions of first order linear differential equations ..... 193
§ 4. Some lemmas ..... 195
§ 5. Proof of the theorems ..... 198
Remarks ..... 205

*Chapter 6. Algebraic independence of the values of E-functions which satisfy second order linear differential equations* ..... 207

§ 1. A general theorem on algebraic independence of the values of an E-function and its derivative ..... 207
§ 2. The functions $K_\lambda(z)$ associated to Bessel functions ..... 209
§ 3. The functions $K_\lambda(z)$ and $e^z$ ..... 219
§ 4. Kummer functions ..... 221
§ 5. Solutions of non-homogeneous linear differential equations ..... 227
Remarks ..... 231

*Chapter 7. Solutions of certain linear differential equations of arbitrary order* ..... 232

§ 1. Solutions of non-homogeneous differential equations ..... 232
§ 2. Solutions of homogeneous differential equations ..... 239
§ 3. Corollaries of Theorems 1 and 2 ..... 244
Remarks ..... 247

*Chapter 8. Arithmetic methods applied to solutions of linear differential equations of arbitrary order* ..... 249

§ 1. Statement of the theorems ..... 249
§ 2. Auxiliary lemmas ..... 252
§ 3. Proof of Theorems 1–5 ..... 258
§ 4. Proof of Theorems 6 and 7 ..... 262
§ 5. Further results ..... 264

*Chapter 9. Siegel's Theorem* ..... 271

§ 1. Statement of the theorem and some basic auxiliary results ..... 271
§ 2. Some lemmas ..... 273
§ 3. Some properties of solutions of second order homogeneous linear differential equations ..... 278
§ 4. Algebraic independence of solutions of a set of second order homogeneous linear differential equations ..... 283
§ 5. Proof of Siegel's Theorem ..... 287
§ 6. Solutions of non-homogeneous linear differential equations ..... 291
§ 7. Generalizations of Siegel's Theorem ..... 300

*Chapter 10. Solutions of linear differential equations of prime order p* ..... 307

§ 1. Statement of the basic results ..... 307
§ 2. The homogeneous ideal $\mathcal{I}$ ..... 313
§ 3. Algebraic functions of several variables ..... 315
§ 4. The differential operator $G$ ..... 319
§ 5. The differential operators $S$ and $\delta$ ..... 322
§ 6. A lemma on linear approximation ..... 325
§ 7. End of the proof of Theorem 7 ..... 328
§ 8. Linear reducibility ..... 332
§ 9. Proof of Theorems 6 and 5 ..... 342
Remarks ..... 347

*Chapter 11. The algebraic independence measure of values of IE-functions* ..... 349

§ 1. Definition of the measures ..... 349
§ 2. The linear independence measure of values of IE-functions ..... 356
§ 3. The algebraic independence measure of values of IE-functions which are not connected by algebraic equations over $\mathbf{C}(z)$ ..... 360
§ 4. Auxiliary results ..... 365

§ 5. The algebraic independence measure of values of IE-functions which are connected by algebraic equations over $\mathbf{C}(z)$ ..... 370
§ 6. Some applications of the general theorems ..... 373
Remarks ..... 377

*Chapter 12. The algebraic independence measure of values of KE-functions* ..... 378

§ 1. The fundamental lemma ..... 378
§ 2. Bounds for the measures of the values of E-functions which are not connected by algebraic equations over $\mathbf{C}(z)$ ..... 383
§ 3. Bounds for the measures of the values of E-functions which are connected by a single algebraic equation over $\mathbf{C}(z)$ ..... 386
§ 4. Bounds for the measures of the values of E-functions which are connected by arbitrary algebraic equations over $\mathbf{C}(z)$ ..... 389
§ 5. Algebraic independence of the values of E-functions in conjugate fields ..... 391
§ 6. An auxiliary theorem ..... 392
§ 7. Consequences of the auxiliary theorem ..... 398
§ 8. Some applications of the general theorems ..... 400
Remarks ..... 404

*Chapter 13. Effective bounds for measures* ..... 405

§ 1. Definitions and notation ..... 405
§ 2. Refinement of the fundamental lemmas ..... 409
§ 3. Bounds for linear independence measures ..... 417
§ 4. Bounds for algebraic independence measures ..... 421
§ 5. Some applications of the general theorems ..... 428
Remarks ..... 434

*Concluding remarks* ..... 436

*Supplementary remarks on recent work for the English edition* ..... 444

*Bibliography* ..... 452