TABLE OF CONTENTS

PREFACE xi

PART I: CLASSICAL GROUPS AND NORMED ALGEBRAS

1. CLASSICAL GROUPS I 3
   - The General Linear Groups 4
   - Groups Defined by Bilinear Forms 6
   - Other Miscellaneous Groups 8
   - Isomorphisms 13
   - Summary 15
2. THE EIGHT TYPES OF INNER PRODUCT SPACES 19
   - The Standard Models 22
   - Orthogonality 23
   - A Canonical Form (The Basis Theorem) 29
   - The Parts of an Inner Product 31
3. CLASSICAL GROUPS II 41
   - Group Representations and Orbits 41
   - Generalized Spheres 42
   - The Basis Theorem Revisited 46
   - Adjoints 47
   - Lie Algebras 48
4. EUCLIDEAN/LORENTZIAN VECTORS SPACES 57
   - The Cauchy–Schwarz Equality 58
   - Special Relativity 62
   - The Cartan–Dieudonné Theorem 67
   - Grassmannians and $\mathrm{SO}^{\uparrow}(p,q)$, the Reduced Special Orthogonal Groups 75
5. DIFFERENTIAL GEOMETRY 81
   - Real n-Manifolds: The Group $\mathrm{GL}(n,\mathbf{R})$ 81
   - Oriented Real n-Manifolds: The Group $\mathrm{GL}^{+}(n,\mathbf{R})$ 82
   - Complex (and Almost Complex) n-Manifolds: The Group $\mathrm{GL}(n,\mathbf{C})$ 83
   - Quaternionic (and Almost Quaternionic) n-Manifolds: The Group $\mathrm{GL}(n,\mathbf{H})\cdot\mathbf{H}^{*}$ 85
   - Manifolds with Volume: The Groups $\mathrm{SL}(n,\mathbf{R})$ and $\mathrm{SL}(n,\mathbf{C})$ . . 86
   - Riemannian Manifolds (of Signature p,q): The Group $O(p,q)$. 87
   - Conformal Manifolds (of Signature p,q): The Group $\mathrm{CO}(p,q)$. 87

5. DIFFERENTIAL GEOMETRY (continued)

Real Symplectic Manifolds: The Group $\mathrm{Sp}(n, \mathbf{R})$ 87
Complex Riemannian Manifolds: The Group $O(n, \mathbf{C})$ 88
Complex Symplectic Manifolds: The Group $\mathrm{Sp}(n, \mathbf{C})$ 88
Kähler Manifolds (of Signature p, q): The Group $U(p, q)$ 89
Special Kähler Manifolds (of Signature p, q): The Group $\mathrm{SU}(p, q)$ 90
HyperKähler Manifolds (of Signature p, q): The Group $\mathrm{HU}(p, q)$ 91
Quaternionic Kähler Manifolds (of Signature p, q):
The Group $\mathrm{HU}(p, q) \cdot \mathrm{HU}(1)$ 92
Quaternionic Skew Hermitian Manifolds:
The Group $\mathrm{SK}(n, \mathbf{H})$ 93
Coincidences of Geometries in Low Dimensions 94

6. NORMED ALGEBRAS 101

The Cayley–Dickson Process 104
The Hurwitz Theorem 107
Cross Products 110
The Exceptional Lie Group G_2 113

7. CALIBRATIONS 125

The Fundamental Theorem 125
The Kähler Case 127
The Special Lagrangian Calibration 130
The Special Lagrangian Differential Equation 134
Examples of Special Lagrangian Submanifolds 138
Associative Geometry 144
The Angle Theorem 146
Generalized Nance Calibrations and Complex Structures . . 156

8. MATRIX ALGEBRAS 163

Representations 163
Uniqueness of Intertwining Operators 167
Automorphisms 169
Inner Products 170

PART II: SPINORS

9. THE CLIFFORD ALGEBRAS 177

The Clifford Automorphisms 181
The Clifford Involutions 182
The Clifford Inner Product 184

9. THE CLIFFORD ALGEBRAS (continued)
The Main Symmetry 185
The Clifford Center 187
Self Duality 188
Trace 189
The Complex Clifford Algebras 190

10. THE GROUPS SPIN AND PIN 195
The Grassmannians and Reflections 197
Additional Groups for Nondefinite Signature 199
The Conformal Pin (or Clifford) Group 201
Determinants 203

11. THE CLIFFORD ALGEBRAS $\mathrm{Cl}(r,s)$ AS ALGEBRAS 207
The Pinor Representations 210
The Spinor Representations 213
The First Proof 215
The Spinor Structure Map on $\mathbf{P}(r,s)$ 217
Even Dimensions 217
Odd Dimensions 220

12. THE SPLIT CASE $\mathrm{Cl}(p,p)$ 227
A Model for $\mathrm{Cl}(p,p)$ 227
Pinor Inner Products for $\mathrm{Cl}(p,p) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P}(p,p))$ 230
The Complex Clifford Algebras
(Continued from Chapter 9) 233
$\mathrm{Cl}(r,s)(r+s \equiv 2p)$ as a Subalgebra of
$\mathrm{Cl}_{\mathbf{C}}(2p) \equiv \mathrm{Cl}(p,p) \otimes_{\mathbf{R}} \mathbf{C}$ 235
The Pinor Reality Map 237
A Second Proof of the Classification Theorem 240
Pure Spinors 241

13. INNER PRODUCTS ON THE SPACES OF SPINORS
AND PINORS 247
The Spinor Inner Product 247
The Spin Representation and the (Reduced) Classical
Companion Group $\mathrm{Cp}^0(r,s)$ 250
The Pinor Inner Products $\check{\varepsilon}$ and $\hat{\varepsilon}$ 252
Pinor Multiplication 266
Signature 268

14. LOW DIMENSIONS 271
Cartan's Isomorphisms 271
Triality 275
Transitive Actions on Spheres 283
The Cayley Plane and the Exceptional Group F_4 289
Clifford Algebras in Low Dimensions 298
Squares of Spinors and Calibrations 308

REFERENCES 315

INDEX 317