

# Contents

Preface .....	v
<b>Chapter 1</b>	
<b>Sphere Packings and Kissing Numbers</b>	
<i>J.H. Conway and N.J.A. Sloane</i> .....	1
1. The Sphere Packing Problem .....	1
1.1 Packing Ball Bearings .....	1
1.2 Lattice Packings .....	3
1.3 Nonlattice Packings .....	7
1.4 $n$ -Dimensional Packings .....	8
1.5 Sphere Packing Problem—Summary of Results .....	12
2. The Kissing Number Problem .....	21
2.1 The Problem of the Thirteen Spheres .....	21
2.2 Kissing Numbers in Other Dimensions .....	21
2.3 Spherical Codes .....	24
2.4 The Construction of Spherical Codes from Sphere Packing .....	26
2.5 The Construction of Spherical Codes from Binary Codes ...	26
2.6 Bounds on $A(n, \phi)$ .....	27
Appendix: Planetary Perturbations .....	29
<b>Chapter 2</b>	
<b>Coverings, Lattices and Quantizers</b>	
<i>J.H. Conway and N.J.A. Sloane</i> .....	31
1. The Covering Problem .....	31
1.1 Covering Space with Overlapping Spheres .....	31
1.2 The Covering Radius and the Voronoi Cells .....	33
1.3 Covering Problem—Summary of Results .....	36
1.4 Computational Difficulties in Packings and Coverings .....	40

2.	Lattices, Quadratic Forms and Number Theory .....	41
2.1	The Norm of a Vector .....	41
2.2	Quadratic Forms Associated with a Lattice .....	42
2.3	Theta Series and Connections with Number Theory .....	44
2.4	Integral Lattices and Quadratic Forms .....	47
2.5	Modular Forms .....	50
2.6	Complex and Quaternionic Lattices .....	52
3.	Quantizers .....	56
3.1	Quantization, Analog-to-Digital Conversion and Data Compression .....	56
3.2	The Quantizer Problem .....	59
3.3	Quantizer Problem—Summary of Results .....	59

### Chapter 3

#### Codes, Designs and Groups

<i>J.H. Conway and N.J.A. Sloane</i> .....	63	
1.	The Channel Coding Problem .....	63
1.1	The Sampling Theorem .....	63
1.2	Shannon's Theorem .....	66
1.3	Error Probability .....	69
1.4	Lattice Codes for the Gaussian Channel .....	71
2.	Error-Correcting Codes .....	75
2.1	The Error-Correcting Code Problem .....	75
2.2	Further Definitions from Coding Theory .....	77
2.3	Repetition, Even Weight and Other Simple Codes .....	79
2.4	Cyclic Codes .....	79
2.5	BCH and Reed-Solomon Codes .....	81
2.6	Justesen Codes .....	82
2.7	Reed-Muller Codes .....	83
2.8	Quadratic Residue Codes .....	84
2.9	Perfect Codes .....	85
2.10	The Pless Double Circulant Codes .....	86
2.11	Goppa Codes and Codes from Algebraic Curves .....	87
2.12	Nonlinear Codes .....	87
2.13	Hadamard Matrices .....	87
3.	$t$ -Designs, Steiner Systems and Spherical $t$ -Designs .....	88
3.1	$t$ -Designs and Steiner Systems .....	88
3.2	Spherical $t$ -Designs .....	89
4.	The Connections with Group Theory .....	90
4.1	The Automorphism Group of a Lattice .....	90
4.2	Constructing Lattices and Codes from Groups .....	92

### Chapter 4

#### Certain Important Lattices and Their Properties

<i>J.H. Conway and N.J.A. Sloane</i> .....	94	
1.	Introduction .....	94
2.	Reflection Groups and Root Lattices .....	95
3.	Gluing Theory .....	99

4.	Notation; Theta Functions .....	101
4.1	Jacobi Theta Functions .....	102
5.	The $n$ -Dimensional Cubic Lattice $Z^n$ .....	106
6.	The $n$ -Dimensional Lattices $A_n$ and $A_n^*$ .....	108
6.1	The Lattice $A_n$ .....	108
6.2	The Hexagonal Lattice .....	110
6.3	The Face-Centered Cubic Lattice .....	112
6.4	The Tetrahedral or Diamond Packing .....	113
6.5	The Hexagonal Close-Packing .....	113
6.6	The Dual Lattice $A_n^*$ .....	115
6.7	The Body-Centered Cubic Lattice .....	116
7.	The $n$ -Dimensional Lattices $D_n$ and $D_n^*$ .....	117
7.1	The Lattice $D_n$ .....	117
7.2	The Four-Dimensional Lattice $D_4$ .....	118
7.3	The Packing $D_n^+$ .....	119
7.4	The Dual Lattice $D_n^*$ .....	120
8.	The Lattices $E_6$ , $E_7$ and $E_8$ .....	120
8.1	The 8-Dimensional Lattice $E_8$ .....	120
8.2	The 7-Dimensional Lattices $E_7$ and $E_7^*$ .....	124
8.3	The 6-Dimensional Lattices $E_6$ and $E_6^*$ .....	125
9.	The 12-Dimensional Coxeter-Todd Lattice $K_{12}$ .....	127
10.	The 16-Dimensional Barnes-Wall Lattice $\Lambda_{16}$ .....	129
11.	The 24-Dimensional Leech Lattice $\Lambda_{24}$ .....	131

Chapter 5

Sphere Packing and Error-Correcting Codes

<i>J. Leech and N.J.A. Sloane</i> .....	136
---	-----

1.	Introduction .....	136
1.1	The Coordinate Array of a Point .....	137
2.	Construction A .....	137
2.1	The Construction .....	137
2.2	Center Density .....	137
2.3	Kissing Numbers .....	138
2.4	Dimensions 3 to 6 .....	138
2.5	Dimensions 7 and 8 .....	138
2.6	Dimensions 9 to 12 .....	139
2.7	Comparison of Lattice and Nonlattice Packings .....	140
3.	Construction B .....	141
3.1	The Construction .....	141
3.2	Center Density and Kissing Numbers .....	141
3.3	Dimensions 8, 9 and 12 .....	142
3.4	Dimensions 15 to 24 .....	142
4.	Packings Built Up by Layers .....	142
4.1	Packing by Layers .....	142
4.2	Dimensions 4 to 7 .....	144
4.3	Dimensions 11 and 13 to 15 .....	144
4.4	Density Doubling and the Leech Lattice $\Lambda_{24}$ .....	145
4.5	Cross Sections of $\Lambda_{24}$ .....	145
5.	Other Constructions from Codes .....	146

5.1	A Code of Length 40 .....	146
5.2	A Lattice Packing in $\mathbf{R}^{40}$ .....	147
5.3	Cross Sections of $\Lambda_{40}$ .....	148
5.4	Packings Based on Ternary Codes .....	148
5.5	Packings Obtained from the Pless Codes .....	148
5.6	Packings Obtained from Quadratic Residue Codes .....	149
5.7	Density Doubling in $\mathbf{R}^{24}$ and $\mathbf{R}^{48}$ .....	149
6.	Construction C .....	150
6.1	The Construction .....	150
6.2	Distance Between Centers .....	150
6.3	Center Density .....	150
6.4	Kissing Numbers .....	151
6.5	Packings Obtained from Reed-Muller Codes .....	151
6.6	Packings Obtained from BCH and Other Codes .....	152
6.7	Density of BCH Packings .....	153
6.8	Packings Obtained from Justesen Codes .....	155
Chapter 6		
Laminated Lattices		
<i>J.H. Conway and N.J.A. Sloane</i> .....		157
1.	Introduction .....	157
2.	The Main Results .....	163
3.	Properties of $\Lambda_0$ to $\Lambda_8$ .....	168
4.	Dimensions 9 to 16 .....	170
5.	The Deep Holes in $\Lambda_{16}$ .....	174
6.	Dimensions 17 to 24 .....	176
7.	Dimensions 25 to 48 .....	177
	Appendix: The Best Integral Lattices Known .....	179
Chapter 7		
Further Connections Between Codes and Lattices		
<i>N.J.A. Sloane</i> .....		181
1.	Introduction .....	181
2.	Construction A .....	182
3.	Self-Dual (or Type I) Codes and Lattices .....	185
4.	Extremal Type I Codes and Lattices .....	189
5.	Construction B .....	191
6.	Type II Codes and Lattices .....	191
7.	Extremal Type II Codes and Lattices .....	193
8.	Constructions A and B for Complex Lattices .....	197
9.	Self-Dual Nonbinary Codes and Complex Lattices .....	202
10.	Extremal Nonbinary Codes and Complex Lattices .....	205
Chapter 8		
Algebraic Constructions for Lattices		
<i>J.H. Conway and N.J.A. Sloane</i> .....		206
1.	Introduction .....	206
2.	The Icosians and the Leech Lattice .....	207

2.1	The Icosian Group .....	207
2.2	The Icosian and Turyn-Type Constructions for the Leech Lattice .....	210
3.	A General Setting for Construction A, and Quebbemann's 64-Dimensional Lattice .....	211
4.	Lattices Over $\mathbb{Z}[e^{i\pi/4}]$ , and Quebbemann's 32-Dimensional Lattice .....	215
5.	McKay's 40-Dimensional Extremal Lattice .....	221
6.	Repeated Differences and Craig's Lattices .....	222
7.	Lattices from Algebraic Number Theory .....	224
7.1	Introduction .....	224
7.2	Lattices from the Trace Norm .....	224
7.3	Examples from Cyclotomic Fields .....	227
7.4	Lattices from Class Field Towers .....	227
7.5	Unimodular Lattices with an Automorphism of Prime Order .....	229
8.	Constructions D and D' .....	232
8.1	Construction D .....	232
8.2	Examples .....	233
8.3	Construction D' .....	235
9.	Construction E .....	236
10.	Examples of Construction E .....	238

## Chapter 9

### Bounds for Codes and Sphere Packings

<i>N.J.A. Sloane</i> .....	245	
1.	Introduction .....	245
2.	Zonal Spherical Functions .....	249
2.1	The 2-Point-Homogeneous Spaces .....	250
2.2	Representations of $G$ .....	252
2.3	Zonal Spherical Functions .....	253
2.4	Positive-Definite Degenerate Kernels .....	256
3.	The Linear Programming Bounds .....	257
3.1	Codes and Their Distance Distributions .....	257
3.2	The Linear Programming Bounds .....	258
3.3	Bounds for Error-Correcting Codes .....	260
3.4	Bounds for Constant-Weight Codes .....	263
3.5	Bounds for Spherical Codes and Sphere Packings .....	263
4.	Other Bounds .....	265

## Chapter 10

### Three Lectures on Exceptional Groups

<i>J.H. Conway</i> .....	267	
1.	First Lecture .....	267
1.1	Some Exceptional Behavior of the Groups $L_n(q)$ .....	267
1.2	The Case $p = 3$ .....	269
1.3	The Case $p = 5$ .....	269
1.4	The Case $p = 7$ .....	269

1.5	The Case $p = 11$ .....	271
1.6	A Presentation for $M_{12}$ .....	273
1.7	Janko's Group of Order 175560 .....	273
2.	Second Lecture .....	274
2.1	The Mathieu Group $M_{24}$ .....	274
2.2	The Stabilizer of an Octad .....	276
2.3	The Structure of the Golay Code $\mathcal{C}_{24}$ .....	278
2.4	The Structure of $P(\Omega)/\mathcal{C}_{24}$ .....	278
2.5	The Maximal Subgroups of $M_{24}$ .....	279
2.6	The Structure of $P(\Omega)$ .....	283
3.	Third Lecture .....	286
3.1	The Group $Co_0 = \cdot 0$ and Some of its Subgroups .....	286
3.2	The Geometry of the Leech Lattice .....	286
3.3	The Group $\cdot 0$ and its Subgroup $N$ .....	287
3.4	Subgroups of $\cdot 0$ .....	290
3.5	The Higman-Sims and McLaughlin Groups .....	292
3.6	The Group $Co_3 = \cdot 3$ .....	293
3.7	Involutions in $\cdot 0$ .....	294
3.8	Congruences for Theta Series .....	294
3.9	A Connection Between $\cdot 0$ and Fischer's Group $Fi_{24}$ .....	295
	Appendix: On the Exceptional Simple Groups .....	296
Chapter 11		
The Golay Codes and the Mathieu Groups		
<i>J.H. Conway</i>	.....	299
1.	Introduction .....	299
2.	Definitions of the Hexacode .....	300
3.	Justification of a Hexacodeword .....	302
4.	Completing a Hexacodeword .....	302
5.	The Golay Code $\mathcal{C}_{24}$ and the MOG .....	303
6.	Completing Octads from 5 of their Points .....	305
7.	The Maximal Subgroups of $M_{24}$ .....	307
8.	The Projective Subgroup $L_2(23)$ .....	308
9.	The Sextet Group $2^6:3:S_6$ .....	309
10.	The Octad Group $2^4:A_8$ .....	311
11.	The Triad Group and the Projective Plane of Order 4 .....	314
12.	The Trio Group $2^6:(S_3 \times L_2(7))$ .....	316
13.	The Octern Group .....	318
14.	The Mathieu Group $M_{23}$ .....	319
15.	The Group $M_{22}:2$ .....	319
16.	The Group $M_{12}$ , the Tetracode and the MINIMOG .....	320
17.	Playing Cards and Other Games .....	323
18.	Further Constructions for $M_{12}$ .....	327
Chapter 12		
A Characterization of the Leech Lattice		
<i>J.H. Conway</i>	.....	331

## Chapter 13

## Bounds on Kissing Numbers

<i>A.M. Odlyzko and N.J.A. Sloane</i> .....	337
1. A General Upper Bound .....	337
2. Numerical Results .....	338

## Chapter 14

## Uniqueness of Certain Spherical Codes

<i>E. Bannai and N.J.A. Sloane</i> .....	340
1. Introduction .....	340
2. Uniqueness of the Code of Size 240 in $\Omega_8$ .....	342
3. Uniqueness of the Code of Size 56 in $\Omega_7$ .....	344
4. Uniqueness of the Code of Size 196560 in $\Omega_{24}$ .....	345
5. Uniqueness of the Code of Size 4600 in $\Omega_{23}$ .....	349

## Chapter 15

## On the Classification of Integral Quadratic Forms

<i>J.H. Conway and N.J.A. Sloane</i> .....	352
1. Introduction .....	352
2. Definitions .....	354
2.1 Quadratic Forms .....	354
2.2 Forms and Lattices; Integral Equivalence .....	355
3. The Classification of Binary Quadratic Forms .....	356
3.1 Cycles of Reduced Forms .....	356
3.2 Definite Binary Forms .....	357
3.3 Indefinite Binary Forms .....	359
3.4 Composition of Binary Forms .....	364
3.5 Genera and Spinor Genera for Binary Forms .....	366
4. The $p$ -Adic Numbers .....	366
4.1 The $p$ -Adic Numbers .....	367
4.2 $p$ -Adic Square Classes .....	367
4.3 An Extended Jacobi-Legendre Symbol .....	368
4.4 Diagonalization of Quadratic Forms .....	369
5. Rational Invariants of Quadratic Forms .....	370
5.1 Invariants and the Oddity Formula .....	370
5.2 Existence of Rational Forms with Prescribed Invariants .....	372
5.3 The Conventional Form of the Hasse-Minkowski Invariant .....	373
6. The Invariance and Completeness of the Rational Invariants .....	373
6.1 The $p$ -Adic Invariants for Binary Forms .....	373
6.2 The $p$ -Adic Invariants for $n$ -Ary Forms .....	375
6.3 The Proof of Theorem 7 .....	377
7. The Genus and its Invariants .....	378
7.1 $p$ -Adic Invariants .....	378
7.2 The $p$ -Adic Symbol for a Form .....	379

7.3	2-Adic Invariants	380
7.4	The 2-Adic Symbol	380
7.5	Equivalences Between Jordan Decompositions	381
7.6	A Canonical 2-Adic Symbol	382
7.7	Existence of Forms with Prescribed Invariants	382
7.8	A Symbol for the Genus	384
8.	Classification of Forms of Small Determinant and of $p$ -Elementary Forms	385
8.1	Forms of Small Determinant	385
8.2	$p$ -Elementary Forms	386
9.	The Spinor Genus	388
9.1	Introduction	388
9.2	The Spinor Genus	389
9.3	Identifying the Spinor Kernel	390
9.4	Naming the Spinor Operators for the Genus of $f$	390
9.5	Computing the Spinor Kernel from the $p$ -Adic Symbols	391
9.6	Tractable and Irrelevant Primes	392
9.7	When is There Only One Class in the Genus?	393
10.	The Classification of Positive Definite Forms	396
10.1	Minkowski Reduction	396
10.2	The Kneser Gluing Method	399
10.3	Positive Definite Forms of Determinant 2 and 3	399
11.	Computational Complexity	402

## Chapter 16

### Enumeration of Unimodular Lattices

<i>J.H. Conway and N.J.A. Sloane</i>	406
1. The Niemeier Lattices and the Leech Lattice	406
2. The Mass Formulae for Lattices	408
3. Verifications of Niemeier's List	410
4. The Enumeration of Unimodular Lattices in Dimensions $n \leq 23$	413

## Chapter 17

### The 24-Dimensional Odd Unimodular Lattices

<i>R.E. Borcherds</i>	421
-----------------------	-----

## Chapter 18

### Even Unimodular 24-Dimensional Lattices

<i>B.B. Venkov</i>	427
1. Introduction	427
2. Possible Configurations of Minimal Vectors	428
3. On Lattices with Root Systems of Maximal Rank	431
4. Construction of the Niemeier Lattices	434
5. A Characterization of the Leech Lattice	437



## Chapter 19

## Enumeration of Extremal Self-Dual Lattices

<i>J.H. Conway, A.M. Odlyzko and N.J.A. Sloane</i> .....	439
1. Dimensions 1-16 .....	439
2. Dimensions 17-47 .....	439
3. Dimensions $n \geq 48$ .....	441

## Chapter 20

## Finding the Closest Lattice Point

<i>J.H. Conway and N.J.A. Sloane</i> .....	443
1. Introduction .....	443
2. The Lattices $Z^n$ , $D_n$ and $A_n$ .....	444
3. Decoding Unions of Cosets .....	446
4. "Soft Decision" Decoding for Binary Codes .....	447
5. Decoding Lattices Obtained from Construction A .....	448
6. Decoding $E_8$ .....	448

## Chapter 21

## Voronoi Cells of Lattices and Quantization Errors

<i>J.H. Conway and N.J.A. Sloane</i> .....	449
1. Introduction .....	449
2. Second Moments of Polytopes .....	451
2.A Dirichlet's Integral .....	451
2.B Generalized Octahedron or Crosspolytope .....	452
2.C The $n$ -Sphere .....	452
2.D $n$ -Dimensional Simplices .....	452
2.E Regular Simplex .....	453
2.F Volume and Second Moment of a Polytope in Terms of its Faces .....	453
2.G Truncated Octahedron .....	454
2.H Second Moment of Regular Polytopes .....	454
2.I Regular Polygons .....	455
2.J Icosahedron and Dodecahedron .....	455
2.K The Exceptional 4-Dimensional Polytopes .....	455
3. Voronoi Cells and the Mean Squared Error of Lattice Quantizers .....	456
3.A The Voronoi Cell of a Root Lattice .....	456
3.B Voronoi Cell for $A_n$ .....	459
3.C Voronoi Cell for $D_n$ ( $n \geq 4$ ) .....	462
3.D Voronoi Cells for $E_6, E_7, E_8$ .....	462
3.E Voronoi Cell for $D_n^*$ .....	463
3.F Voronoi Cell for $A_n^*$ .....	472
3.G The Walls of the Voronoi Cell .....	474

## Chapter 22

## A Bound for the Covering Radius of the Leech Lattice

<i>S.P. Norton</i> .....	476
--------------------------	-----

## Chapter 23

## The Covering Radius of the Leech Lattice

<i>J.H. Conway, R.A. Parker and N.J.A. Sloane</i> .....	478
1. Introduction .....	478
2. The Coxeter-Dynkin Diagram of a Hole .....	480
3. Holes Whose Diagram Contains an $A_n$ Subgraph .....	484
4. Holes Whose Diagram Contains a $D_n$ Subgraph .....	495
5. Holes Whose Diagram Contains an $E_n$ Subgraph .....	502

## Chapter 24

## Twenty-Three Constructions for the Leech Lattice

<i>J.H. Conway and N.J.A. Sloane</i> .....	506
1. The "Holy Constructions" .....	506
2. The Environs of a Deep Hole .....	510

## Chapter 25

## The Cellular Structure of the Leech Lattice

<i>R.E. Borcherds, J.H. Conway and L. Queen</i> .....	513
1. Introduction .....	513
2. Names for the Holes .....	513
3. The Volume Formula .....	514
4. The Enumeration of the Small Holes .....	519

## Chapter 26

## Lorentzian Forms for the Leech Lattice

<i>J.H. Conway and N.J.A. Sloane</i> .....	522
1. The Unimodular Lorentzian Lattices .....	522
2. Lorentzian Constructions for the Leech Lattice .....	523

## Chapter 27

## The Automorphism Group of the 26-Dimensional Even Unimodular Lorentzian Lattice

<i>J.H. Conway</i> .....	527
1. Introduction .....	527
2. The Main Theorem .....	528

## Chapter 28

## Leech Roots and Vinberg Groups

<i>J.H. Conway and N.J.A. Sloane</i> .....	532
1. The Leech Roots .....	532
2. Enumeration of the Leech Roots .....	541
3. The Lattices $I_{n,1}$ for $n \leq 19$ .....	547
4. Vinberg's Algorithm and the Initial Batches of Fundamental Roots .....	547
5. The Later Batches of Fundamental Roots .....	550

Chapter 29

The Monster Group and its 196884-Dimensional Space

*J.H. Conway* ..... 554

1. Introduction ..... 554
2. The Golay Code  $\mathcal{C}$  and the Parker Loop  $\mathcal{P}$  ..... 556
3. The Mathieu Group  $M_{23}$ : the Standard Automorphisms of  $\mathcal{P}$  ..... 556
4. The Golay Cocode  $\mathcal{C}^*$  and the Diagonal Automorphisms ... 556
5. The Group  $N$  of Triple Maps ..... 557
6. The Kernel  $K$  and the Homomorphism  $g \rightarrow \bar{g}$  ..... 557
7. The Structures of Various Subgroups of  $\bar{N}$  ..... 557
8. The Leech Lattice  $\Lambda_{24}$  and the Group  $\bar{Q}_\lambda$  ..... 558
9. Short Elements ..... 559
10. The Basic Representations of  $N_\lambda$  ..... 559
11. The Dictionary ..... 560
12. The Algebra ..... 561
13. The Definition of the Monster Group  $\bar{G}$ , and its Finiteness ..... 561
14. Identifying the Monster ..... 562

Appendix 1. Computing in  $\mathcal{P}$  ..... 563

Appendix 2. A Construction for  $\mathcal{P}$  ..... 563

Appendix 3. Some Relations in  $Q_\lambda$  ..... 564

Appendix 4. Constructing Representations for  $N_\lambda$  ..... 566

Appendix 5. Building the Group  $\bar{G}$ , ..... 567

Chapter 30

A Monster Lie Algebra?

*R.E. Borcherds, J.H. Conway, L. Queen and*

*N.J.A. Sloane* ..... 568

Bibliography ..... 572

Index ..... 641