

# Contents

Foreword	xi
Preface	xv
Acknowledgments	xvii
Introduction	xix
1. ALGEBRAIC PRELIMINARIES	1
1.1 Coalgebras	1
1.2 The Algebra $C^*$	10
1.3 The Coalgebra $A^\circ$	17
1.3.1 The Construction and Characterizations of $A^\circ$	17
1.3.2 Double Duals	20
1.3.3 The Fundamental Theorem of Coalgebras	21
1.4 Rational Modules and Comodules	23
1.4.1 Rational Modules	23
1.4.2 Comodules	24
1.4.3 $M_r$ and $M^r$	27
1.4.4 $M_r$ Characterized in Terms of Annihilators	27
1.4.5 Another Proof of the Fundamental Theorem of Coalgebras	29
1.5 Bialgebras	32
1.6 Hopf Algebras	39
1.6.1 The Convolution Algebra	40
1.6.2 Definition of Hopf Algebra and Antipode	41
1.7 The Coradical and the Coradical Filtration	46
1.8 Pointed Hopf Algebras	53
1.9 (Co)Module (Co)Algebras	54
1.9.1 $H$ -Module Algebras and Coalgebras	55
1.9.2 $H$ -Comodule Algebras and Coalgebras	59

2. THE QUANTUM YANG-BAXTER EQUATION (QYBE)	65
2.1 The Constant Form of the QYBE	66
2.1.1 The Constant Form of the QYBE in H-S Notation	67
2.1.2 The Constant Form of the QYBE in Coordinates	67
2.2 The Braid Equation	68
2.3 Symmetries	70
2.4 The One-Parameter Form of the QYBE	72
2.5 The Two-Parameter Form of the QYBE	74
2.6 A System of Polynomial Equations (the QYB Variety)	74
2.7 The Bialgebra Associated to the QYBE	76
2.7.1 A Module Action Associated to a QYBE Solution	76
2.7.2 Comodule Coaction	77
2.8 Factoring a QYBE Solution Over a Bialgebra	78
2.9 Compatibility Conditions in the Constant Case	79
2.9.1 The Fundamental Compatibility Condition in Coordinates	79
2.9.2 The (Co)Commutative Compatibility Condition	80
2.9.3 Compatibility Conditions in H-S Notation	81
2.10 Compatibility Conditions in the Parameterized Cases	81
2.11 Reducing the Degree of the QYB Variety	83
2.11.1 From Cubic to Quadratic to Linear	83
2.11.2 A Curious Example	83
3. CATEGORIES OF QUANTUM YANG-BAXTER MODULES	87
3.1 Various Categories	87
3.1.1 Left QYB $A$ -Modules	88
3.1.2 CQYB $A$ -Modules	89
3.1.3 Right QYB $A$ -Modules	90
3.1.4 Weak QYB $A$ -Modules	91
3.2 Congruence in ${}_A QYB$	93
3.3 Recollections of Various Module and Comodule Structures	94
3.4 General Constructions in ${}_A QYB$	96
3.4.1 Sub-Objects, Quotient Objects of ${}_A QYB$	96
3.4.2 Direct Sums in ${}_A QYB$	96
3.4.3 Duals of Objects of ${}_A QYB$	96
3.4.4 Structure Induced from Objects of ${}_A QYB$	99
3.5 Constructions in ${}_H QYB$ when $H^{op}$ has an Antipode	99
3.5.1 Equivalent Formulations of Compatibility	99
3.5.2 The Rational Part of a Left $H, H^*$ -Module	101
3.5.3 Direct Products in ${}_A QYB$	102
3.5.4 Sub-Objects of Objects of ${}_H QYB$ when $H^{op}$ has an Antipode	103
3.6 The Relationship Between QYBE Solutions $R$ and $R^r$	104
3.7 QYB Structures on $H$ when $H^{op}$ is a Hopf Algebra	105

3.7.1	Generalized Coadjoint Action	106
3.7.2	Generalized Adjoint Action	109
3.8	Tensor Product in ${}_A QYB$	110
3.8.1	The Tensor Algebra	113
3.8.2	$\text{Hom}(M, N)$ and Quantum Yang–Baxter Submodules	113
3.9	Tensor Product of Parameterized QYBE Solutions	114
3.10	Algebras of ${}_H QYB$	115
3.11	Coalgebras, Bialgebras, and Hopf Algebras of ${}_H QYB$	116
3.12	Smash Biproducts Associated to ${}_H^H QYB$	117
4.	MORE ON THE BIALGEBRA ASSOCIATED TO THE QYBE	121
4.1	Module–Comodule Compatibility Revisited	121
4.2	A Basis-Free Description of the FRT Construction	128
4.3	$A(R)^{op}$ , $A(R)^{cop}$ , and $A(R)^{op\ cop}$ as FRT Constructions	131
4.4	Conditions for $\widetilde{A(R)}$ to be a Pointed Bialgebra	138
5.	THE FUNDAMENTAL EXAMPLE OF A QUANTUM GROUP	143
5.1	Review of $\text{SL}(2, k)$	143
5.1.1	The Coordinate Ring of $\text{SL}(2, k)$	144
5.1.2	The Lie Algebra $\text{sl}(2, k)$	146
5.1.3	Irreducible Representations of $\text{sl}(2, k)$	148
5.2	Derivations and (Co)Algebra Actions Revisited	149
5.3	A Hopf Algebra Closely Related to $k[\text{SL}(2, k)]$	150
5.4	Grouplikes and Skew Primitives of $k[\text{SL}_q(2, k)]^o$	151
5.5	Embedding $\mathcal{U}(\text{sl}(2, k))$ into $k[\text{SL}(2, k)]^o$	153
5.6	Quantum Analogs of $\mathcal{U}(\text{sl}(2, k))$	155
6.	QUASITRIANGULAR STRUCTURES AND THE DOUBLE	161
6.1	Quasitriangular Algebras	161
6.2	Quasitriangular Structures Arising from Integrals	162
6.3	Quasitriangular Bialgebras and Quasitriangular Hopf Algebras	164
6.4	The Quantum Double	175
6.5	Some Fundamental Examples of Pointed Hopf Algebras	181
6.5.1	Q-Binomial Coefficients	182
6.5.2	Construction of the Examples	184
6.6	A Family of QT Hopf Algebras and Associated Doubles	186
6.6.1	Construction and Properties of $H_{(N, \nu, \omega)}$	187
6.6.2	Construction and Properties of $U_{(N, \nu, \omega)}$	191
7.	COQUASITRIANGULAR STRUCTURES	197
7.1	Further Properties of $A(R)$	197

7.2	Coquasitriangular Coalgebras	199
7.3	Coquasitriangular Bialgebras and Hopf Algebras	203
7.4	The Free Coquasitriangular Bialgebra	209
7.5	One-Parameter QYBE, Coquasitriangularity, and Tensor Product	213
7.5.1	$R$ -Commutative Spectral Parameter	214
7.5.2	Constructions when $X$ is a Group	215
7.5.3	Tensor Product of One-Parameter QYBE Solutions	218
8.	SOME CLASSES OF SOLUTIONS	219
8.1	Some Consequences of $M$ -Reduction	220
8.2	When $\widetilde{A}(R)$ is Generated by Grouplike Elements	222
8.3	Solutions when $\text{Dim}M = 2$ and $\widetilde{A}(R)$ is Pointed	226
8.4	Patching and Solutions in Higher Dimension	232
8.5	A Class of Weak QYB Modules	233
8.6	Some One-Parameter Solutions	244
8.6.1	Some Specific Solutions	244
8.6.2	A $\rho$ -Perturbation Example	247
9.	CATEGORICAL CONSTRUCTIONS	249
9.1	Coends	249
9.2	Quasi-Symmetric Monoidal Categories	250
9.3	Rigid Monoidal Categories and Hopf Algebras	254
9.4	Categories and Coquasitriangular Hopf Algebras	258
9.5	The QYBE in Other Categories	258
9.6	The Category of Graded Modules	259
	Appendices	261
	A–Prerequisites	261
A.1	The Ground Ring $k$ and Basic $k$ -Linear Maps	261
A.2	Algebras, Coalgebras, and Their Representations	262
A.3	Various Notations Related to the QYBE	263
A.3.1	Structure Constants	263
A.3.2	Heyneman-Sweedler and H-S Notations	267
A.3.3	Categorical Notation	268
A.4	Some Results from Linear Algebra	269
A.4.1	Rank of Tensors and Endomorphisms	269
A.4.2	Closed Subspaces of $U^*$	272
A.4.3	Cofinite Subspaces and Continuous Linear Maps	277
	References	281
	Index	291