Contents

Preface ix

Introduction 1

Chapter 1. The Euclidean World 5
1.1. The Euclidean line and plane 5
Cartesian model of the Euclidean straight line and plane 7
The Euclidean plane and complex numbers 11
Some problems 13
1.2. n-dimensional Euclidean space 17
The vector space $\mathbb{R}^n$ 17
The affine space $\mathbb{A}^n$ and the Euclidean space $\mathbb{E}^n$ 20
1.3. Introduction to the multidimensional world of Euclidean geometry 21
Affine varieties 22
Determinants and volumes 23
Simplices and balls 27
Problems 27

Chapter 2. The Affine World 33
2.1. The affine line and the affine plane 33
Arithmetical model of the affine line 33
Arithmetical model of the affine plane 33
Linear equations on the plane 34
Convex geometry on the plane and the theory of linear inequalities 36
The fundamental theorem of affine geometry 39
2.2. Affine space. Theory of linear equations and inequalities 42
2.3. Introduction to finite-dimensional convex geometry 43
The Carathéodory and Radon lemmas 43
Helly's theorem 44
Problems 45

Chapter 3. The Projective World 47
3.1. The projective line and the projective plane 47
A model and some facts of the geometry of projective line 47
The projective plane 52
Pappus' and Desargues' theorems 53
3.2. Projective n-space 57
Problems 59

Chapter 4. Conics and Quadrics 61

4.1. Plane curves of the second order 61
Metric, affine, and projective classification of second-order curves 61
The ellipse, hyperbola, and parabola 63
The ellipse, hyperbola, and parabola as conic sections 68
4.2. Additional remarks 69
Fourth-degree equations 69
The theorem about the conic passing through five points 69
The theorem about the pencil of conics passing through four points 70
The butterfly problem 70
Hyperbolas with perpendicular asymptotes 70
Pascal's theorem 71
Common chords of two conics inscribed in the same conic 72
4.3. Some properties of quadrics 73
Two families of straight lines on a quadric 75
Problems 77

Chapter 5. The World of Non-Euclidean Geometries 83
5.1. The circle and the two-dimensional sphere: one- and two-dimensional Riemannian geometries 83
The circle and the sphere 83
Elementary spherical geometry 86
Geometry of the n-sphere 88
Riemannian, or elliptic, geometry 88
5.2. Lobachevsky geometry 89
The Klein model of Lobachevsky geometry 89
Linear-fractional transformations and stereographic projections 93
Other models of Lobachevsky geometry 95
Elementary hyperbolic geometry 100
5.3. Isometries in the three geometries 104
Isometries of Euclidean space 105
Isometries of the sphere 110
Three types of proper motions of the Lobachevsky plane 113
Problems 116

Chapter 6. The Infinite-Dimensional World 123
6.1. Basic definitions 123
6.2. Statements of theorems 126
6.3. Proofs of the theorems 128
6.4. Concluding comments 132

Addendum 135
1. Geometry and physics 135
Projectiles move along parabolas (Galileo and Newton) 135
The planets move along ellipses, and the asteroids, along second-order curves (Kepler and Newton) 136
Geometry and special relativity (Einstein and Minkowski) 137
2. Polyhedra and polygons 140
Convex polyhedra 140
The Euler–Poincaré formula for the alternating sum of the numbers of faces (of various dimensions) of a convex polyhedron 140

Dual polyhedra 142
The Gram–Sommerville formula for the alternating sum of solid angles at the faces of a convex polyhedron 143
The Gauss–Bonnet theorem 145
The Minkowski theorem 146
The Cauchy theorem on rigid convex polyhedra 148
Regular polyhedra 151
The Cauchy formula 154
The Steiner–Minkowski formula 155
Polygons in $\mathbb{R}^m$ 156
3. Additional questions of projective geometry 158
The complex projective space $\mathbb{C}P^n$ 158
The polar line of a point with respect to a curve in $\mathbb{C}P^2$ 160
Projective duality 163
Fixed points of projective transformations of the line and Steiner's construction 166
Projective involutions and harmonic quadruples of points and lines 168
Problems 171
4. Special properties of conics and quadrics 174
Confocal conics and quadrics 174
Rational parametrizations of conics 178
Poncelet's theorem and the zigzag theorem 179
The cross ratio of four points on a conic 182
Problems 185
5. Additional topics of non-Euclidean geometries 185
Paving the sphere, the plane, and the Lobachevsky plane by triangles 185
Fundamental domains of the modular group 187
Poincaré's theorem about fundamental polygons 190
The Lobachevsky space 193
The quaternion model 196
About the axiomatic approach to Euclidean and non-Euclidean geometries 196
A brief excursion into the history of non-Euclidean geometry 200
Conic sections in spherical and Lobachevsky geometries 201
Parabolic mirrors in Lobachevsky geometry 205
The volume of a simplex with vertices on the absolute 207
Problems 209

Solutions, Hints, and Answers 211
Chapter 1 211
Chapter 2 218
Chapter 3 219
Chapter 4 221
Chapter 5 230
Addendum 244

Bibliography 251

Author Index 253

Subject Index 255