CONTENTS

Preface to the Second Edition xi
Preface to the First Edition xiii
List of Figures xvi
Prerequisites by Chapter xvii
Standard Notation xviii

INTRODUCTION: CLOSED LINEAR GROUPS 1

1. Linear Lie Algebra of a Closed Linear Group 1
2. Exponential of a Matrix 6
3. Closed Linear Groups 9
4. Closed Linear Groups as Lie Groups 11
5. Homomorphisms 16
6. Problems 20

I. LIE ALGEBRAS AND LIE GROUPS 23

1. Definitions and Examples 24
2. Ideals 29
3. Field Extensions and the Killing Form 33
4. Semidirect Products of Lie Algebras 38
5. Solvable Lie Algebras and Lie's Theorem 40
6. Nilpotent Lie Algebras and Engel's Theorem 45
7. Cartan's Criterion for Semisimplicity 49
8. Examples of Semisimple Lie Algebras 56
9. Representations of $\mathfrak{sl}(2, \mathbb{C})$ 62
10. Elementary Theory of Lie Groups 68
11. Covering Groups 81
12. Complex Structures 91
13. Aside on Real-analytic Structures 98
14. Automorphisms and Derivations 100
15. Semidirect Products of Lie Groups 102
16. Nilpotent Lie Groups 106
17. Classical Semisimple Lie Groups 110
18. Problems 118

II. COMPLEX SEMISIMPLE LIE ALGEBRAS 123

1. Classical Root-space Decompositions 124
2. Existence of Cartan Subalgebras 129
3. Uniqueness of Cartan Subalgebras 137
4. Roots 140
5. Abstract Root Systems 149
6. Weyl Group 162
7. Classification of Abstract Cartan Matrices 170
8. Classification of Nonreduced Abstract Root Systems 184
9. Serre Relations 186
10. Isomorphism Theorem 196
11. Existence Theorem 199
12. Problems 203

III. UNIVERSAL ENVELOPING ALGEBRA 213

1. Universal Mapping Property 213
2. Poincaré–Birkhoff–Witt Theorem 217
3. Associated Graded Algebra 222
4. Free Lie Algebras 228
5. Problems 229

IV. COMPACT LIE GROUPS 233

1. Examples of Representations 233
2. Abstract Representation Theory 238
3. Peter–Weyl Theorem 243
4. Compact Lie Algebras 248
5. Centralizers of Tori 251
6. Analytic Weyl Group 260
7. Integral Forms 264
8. Weyl's Theorem 268
9. Problems 269

V. FINITE-DIMENSIONAL REPRESENTATIONS 273

1. Weights 274
2. Theorem of the Highest Weight 279
3. Verma Modules 283
4. Complete Reducibility 290
5. Harish-Chandra Isomorphism 300

V. FINITE-DIMENSIONAL REPRESENTATIONS
6. Weyl Character Formula 314
7. Parabolic Subalgebras 325
8. Application to Compact Lie Groups 333
9. Problems 339

VI. STRUCTURE THEORY OF SEMISIMPLE GROUPS 347
1. Existence of a Compact Real Form 348
2. Cartan Decomposition on the Lie Algebra Level 354
3. Cartan Decomposition on the Lie Group Level 361
4. Iwasawa Decomposition 368
5. Uniqueness Properties of the Iwasawa Decomposition 378
6. Cartan Subalgebras 384
7. Cayley Transforms 389
8. Vogan Diagrams 397
9. Complexification of a Simple Real Lie Algebra 406
10. Classification of Simple Real Lie Algebras 408
11. Restricted Roots in the Classification 422
12. Problems 426

VII. ADVANCED STRUCTURE THEORY 433
1. Further Properties of Compact Real Forms 434
2. Reductive Lie Groups 446
3. KAK Decomposition 458
4. Bruhat Decomposition 460
5. Structure of M 464
6. Real-rank-one Subgroups 470
7. Parabolic Subgroups 474
8. Cartan Subgroups 487
9. Harish-Chandra Decomposition 499
10. Problems 514

VIII. INTEGRATION 523
1. Differential Forms and Measure Zero 523
2. Haar Measure for Lie Groups 530
3. Decompositions of Haar Measure 535
4. Application to Reductive Lie Groups 539
5. Weyl Integration Formula 547
6. Problems 552

IX. INDUCED REPRESENTATIONS AND BRANCHING THEOREMS 555

1. Infinite-dimensional Representations of Compact Groups 556
2. Induced Representations and Frobenius Reciprocity 563
3. Classical Branching Theorems 568
4. Overview of Branching 571
5. Proofs of Classical Branching Theorems 577
6. Tensor Products and Littlewood–Richardson Coefficients 596
7. Littlewood's Theorems and an Application 602
8. Problems 609

X. PREHOMOGENEOUS VECTOR SPACES 615

1. Definitions and Examples 616
2. Jacobson–Morozov Theorem 620
3. Vinberg's Theorem 626
4. Analysis of Symmetric Tensors 632
5. Problems 638

APPENDICES

A. Tensors, Filtrations, and Gradings
1. Tensor Algebra 639
2. Symmetric Algebra 645
3. Exterior Algebra 651
4. Filtrations and Gradings 654
5. Left Noetherian Rings 656

B. Lie's Third Theorem
1. Levi Decomposition 659
2. Lie's Third Theorem 662
3. Ado's Theorem 662
4. Campbell–Baker–Hausdorff Formula 669

C. Data for Simple Lie Algebras
1. Classical Irreducible Reduced Root Systems 683
2. Exceptional Irreducible Reduced Root Systems 686
3. Classical Noncompact Simple Real Lie Algebras 693
4. Exceptional Noncompact Simple Real Lie Algebras 706

Hints for Solutions of Problems 719
Historical Notes 751
References 783
Index of Notation 799
Index 805