

Contents

Chapter 0. Introduction	1
Chapter 1. Poisson Lie Groups	5
1. Poisson manifolds	5
1.1. Poisson algebras and Poisson manifolds	5
1.2. Symplectic leaves in a Poisson manifold	7
2. Lie bialgebras and Manin triples	10
2.1. Lie bialgebras	10
2.2. Co-Poisson Hopf algebras	12
2.3. Manin triples	14
3. Poisson groups	17
3.1. Poisson affine algebraic groups and Poisson Hopf algebras	17
3.2. Poisson Lie groups	18
3.3. The correspondence between Poisson Lie groups and Lie bialgebras	20
3.4. Symplectic leaves in Poisson Lie groups and dressing action	23
4. Lie bialgebras and classical r -matrices	27
4.1. Coboundary, quasi-triangular and triangular Lie bialgebras	27
4.2. The classification of quasi-triangular Lie bialgebras	30
4.3. Frobenius Lie algebras and CYBE	35
5. Compact Poisson Lie groups	37
5.1. Quasi-triangular compact Poisson Lie groups	37
5.2. Symplectic leaves and Bruhat decomposition	38
5.3. Symplectic leaves in simple complex Poisson Lie groups	42
6. Poisson G -manifolds	43
6.1. Poisson G -manifolds and moment maps	43
6.2. Poisson homogeneous G -manifolds	49
7. Historical remarks	55
Chapter 2. Quantized Universal Enveloping Algebras	57
1. Quantization of Lie bialgebras	57
1.1. Definition of quantization	57
1.2. The quantization of complex simple Lie algebras	58
1.3. Existence and uniqueness of quantization	61
2. QUE-algebras and R -matrices	61
2.1. Types of Hopf algebras	62
2.2. Double Hopf algebras	64
2.3. The quantum double and the universal quantum R -matrix	65
2.4. Twisted version of $U_{\hbar}\mathfrak{g}$	71

3. Center of quasi-triangular Hopf algebras	72
3.1. Two central element constructions	72
3.2. Square of the antipode in the almost-cocommutative case	74
3.3. The quasi-triangular case	76
4. Center of $U_h\mathfrak{g}$ and quantum Harish-Chandra homomorphism	78
4.1. Central elements of $U_h\mathfrak{g}$	79
4.2. Quantum Harish-Chandra homomorphism	80
5. Finite-dimensional $U_h\mathfrak{g}$ -modules	82
5.1. Finite-dimensional modules and highest weights	82
5.2. Central characters and the quantum Harish-Chandra homomorphism	85
6. Tensor products of $U_h\mathfrak{g}$ -modules and tensor categories	86
7. Fixed quantization parameter	89
7.1. The complex Hopf algebra $U_q\mathfrak{g}$	90
7.2. Quasi- R -matrix	90
7.3. Admissible finite-dimensional $U_q\mathfrak{g}$ -modules	92
7.4. Twisted version of $U_q\mathfrak{g}$	93
8. Historical remarks	94
 Chapter 3. Quantized Algebras of Functions	 95
1. Main definitions	95
1.1. Hopf $*$ -algebras	95
1.2. Quantized algebra of regular functions	96
2. Properties of the quantized algebras of functions	97
2.1. Basic properties	97
2.2. Triangular decomposition of $\mathbb{C}[G]_q$	99
2.3. The involution $*$ in $\mathbb{C}[K]_q$	100
3. Examples: $\mathbb{C}[SL_2(\mathbb{C})]_q$ and $\mathbb{C}[SU(2)]_q$	101
4. Representation theory of $\mathbb{C}[SU(2)]_q$	104
4.1. Unitarizable simple $\mathbb{C}[SL_2(\mathbb{C})]_q$ -modules	105
4.2. Irreducible $*$ -representations of $\mathbb{C}[SU(2)]_q$	107
5. Representation theory of $\mathbb{C}[K]_q$	109
5.1. Highest weight modules and primitive ideals	109
5.2. Primitive ideals and Schubert cells	111
5.3. Representations of $\mathbb{C}[K]_q$ and Schubert cells	113
6. Representations of $\mathbb{C}[K]_q$ and symplectic leaves	116
6.1. Elementary representations of $\mathbb{C}[K]_q$	116
6.2. Tensor product theorem	119
7. Representation theory of the twisted algebras of functions	122
7.1. Twisted quantized algebras of functions	122
7.2. Representations of $\mathbb{C}[K^u]_q$ and quantum tori	123
7.3. Representations of $\mathbb{C}[K^u]_q$ and symplectic leaves in K^u	127
7.4. Representations of $\mathbb{C}[K(0, u)]_q$	128
8. Representations of formal quantized algebras of functions	130
9. Historical remarks	131
 Chapter 4. Quantum Weyl Group and the Universal Quantum R -Matrix	 133
1. Motivations: Weyl group in the quasi-classical picture	133

2. Quantum Weyl group: definitions	135
2.1. The case of $\mathbb{C}[SL_2(\mathbb{C})]_{q_i}$	136
2.2. General case	138
2.3. Formal case	139
3. Quantum root vectors	140
3.1. Poincaré–Birkhoff–Witt theorem	140
3.2. Properties of quantum root vectors	141
4. The universal quantum R -matrix	143
4.1. The PBW-basis and the universal quantum R -matrix	143
4.2. The case of fixed q	145
5. Applications to the representations of the braid groups	145
6. Historical remarks	146
Bibliography	149