CONTENTS

Notation xiii

Introduction 1

I LOWER SEMICONTINUITY OF INTEGRAL FUNCTIONALS

1 Lower semicontinuity and coerciveness 11
- 1.1 Lower semicontinuity 11
- 1.2 Yosida transforms 13
- 1.3 Coerciveness conditions. The direct method 16
- 1.4 Exercises 17

2 Weak convergence 18
- 2.1 Weak convergence in Lebesgue spaces 18
- 2.2 Weak convergence in Sobolev spaces 22
- 2.3 Weak* convergence of measures 22
- 2.4 Weak compactness criteria in L^1 24
- 2.5 Exercises 27

3 Minimum problems in Sobolev spaces 28
- 3.1 The direct method. An example of application 28
- 3.2 Borel and Carathéodory functions 29
- 3.3 Rellich's Theorem and equivalent conditions for lower semicontinuity 31
- 3.4 Exercises 32

4 Necessary conditions for weak lower semicontinuity 33
- 4.1 General necessary conditions 33
- 4.2 $W^{1,p}$-quasiconvexity 34
- 4.3 Rank-1-convexity 40
- 4.4 Exercises 41

5 Sufficient conditions for weak lower semicontinuity 42
- 5.1 Convexity 42
- 5.2 Polyconvexity 45
- 5.3 Quasiconvexity 48
- 5.4 Exercises 52

6 The structure of quasiconvex functions 54
- 6.1 Quasiconvexity of polyconvex functions 54
- 6.2 Quasiconvexification 55

6.3 Example of a quasiconvex non-polyconvex function 59
6.4 Example of a rank-1-convex non-quasiconvex function 60

II Γ-CONVERGENCE

7 A naïve introduction to Γ-convergence 65
7.1 Definition and basic properties 65
7.2 Lower and upper Γ-limits 67
7.3 Further properties. Compactness 70
7.4 Exercises 72

8 The indirect methods of Γ-convergence 73
8.1 Γ-limits and Yosida transforms 73
8.2 An example: Γ-limits of quadratic functionals 74

9 Direct methods. An integral representation result 77
9.1 Localization 77
9.2 Integral representation on Sobolev spaces 77
9.3 Integral representation of homogeneous functionals 81

10 Increasing set functions 82
10.1 Increasing set functions 82
10.2 A characterization of measures as set functions 82
10.3 Increasing set functions and compactness of Γ-limits 84

11 The fundamental estimate 85
11.1 Fundamental estimates 85
11.2 Subadditivity of Γ-limits 88
11.3 Γ-limits and boundary values 90
11.4 Exercises 92

12 Integral functionals with standard growth conditions 93
12.1 Standard growth conditions 93
12.2 Fundamental estimate 93
12.3 Compactness for the Γ-limits 95
12.4 Γ-limits of homogeneous functionals 96
12.5 Exercises 98

III BASIC HOMOGENIZATION

13 A 1-dimensional example 101
13.1 The cell-problem homogenization formula 101
13.2 The asymptotic homogenization formula 103
13.3 Proof of the Γ-convergence 104
13.4 Exercises 106

14 Periodic homogenization 108
14.1 The asymptotic homogenization formula 109

14.2 The Homogenization Theorem 111
14.3 Convex homogenization 114
14.3.1 The cell-problem formula 114
14.3.2 Non-coercive convex homogenization 115
14.4 A counterexample to the cell-problem formula 120
14.5 An application: homogenization of elliptic equations in divergence form 123
14.6 Exercises 125

15 Almost-periodic homogenization 128
15.1 Homogenization of uniformly almost-periodic functionals 128
15.2 An example: loss of smoothness by homogenization 135
15.3 Exercises 140

16 Two applications 142
16.1 Homogenization of Riemannian metrics 142
16.2 Homogenization of Hamilton Jacobi equations 145

17 A closure theorem for the homogenization 150
17.1 A closure theorem 150
17.2 An application: homogenization of Besicovitch almost-periodic functionals 156

18 Loss of polyconvexity by homogenization 160
18.1 An example 160

IV FINER HOMOGENIZATION RESULTS

19 Homogenization of connected media 167
19.1 A homogenization theorem on periodic connected domains 167
19.2 Convergence of Neumann boundary value problems 177
19.3 Convergence of Dirichlet boundary value problems 179

20 Homogenization with stiff and soft inclusions 181
20.1 Media with stiff and soft inclusions 181
20.2 The Homogenization Theorem 183
20.3 Convergence of minima 190
20.4 A Lavrentiev phenomenon 193
20.5 Loss of polyconvexity after homogenization 196

21 Homogenization with non-standard growth conditions 199
21.1 A class of non-standard integrals 199
21.2 Convex homogenization 202
21.3 Non-convex homogenization 203
21.4 Exercises 212

22 Iterated homogenization 214
22.1 Statement of the Iterated Homogenization Theorem 214
22.2 Proof of the Iterated Homogenization Theorem 215
22.3 Exercises 222

23 Correctors for the homogenization 227
23.1 Convergence of momenta in homogenization 227
23.2 Definition and some properties of the correctors 234
23.3 Statement and proof of the correctors result 240
23.4 Correctors in the quasiperiodic case 246
23.5 Exercises 248

24 Homogenization of multi-dimensional structures 249
24.1 A smooth approach 249
24.2 A measure Sobolev-space approach 253
24.3 Homogenization of periodic thin structures 263
24.4 Exercises 268

V APPENDICES

A Almost-periodic functions 273

B Construction of extension operators 277

C Some regularity results 287

References 289
Notes to references 294

Index 297