

# Contents

**Preface to the English Edition** **vii**

**Introduction** **xxiii**

**1 Elementary Classes of Relations** **1**

1.1 Local Isomorphisms Between Relations . . . . . 1

1.2 Examples . . . . . 5

1.3 Infinite Back-and-Forth . . . . . 11

1.4 Historic and Bibliographic Notes . . . . . 13

**2 The Language Associated with a Relation** **15**

2.1 Formulas . . . . . 15

2.2 Connections to the Back-and-Forth Technique . . . . . 23

2.3 Models and Theories . . . . . 25

2.4 Elementary Extensions: Tarski's Test,  
Löwenheim's Theorem . . . . . 27

2.5 Historic and Bibliographic Notes . . . . . 29

**3 Extensions of the Language: Structures** **31**

3.1 Multirelations, Relational Structures . . . . . 31

3.2 Functions . . . . . 33

3.3 Löwenheim's Theorem Revisited . . . . . 36

3.4 Historic and Bibliographic Notes . . . . . 37

<b>4</b>	<b>Compactness</b>	<b>38</b>
4.1	Ultraproducts . . . . .	38
4.2	Compactness, Löwenheim-Skolem Theorem, Theorem of Common Elementary Extensions . . . . .	42
4.3	Henkin's Method . . . . .	47
4.4	Historic and Bibliographic Notes . . . . .	52
<b>5</b>	<b>The Back-and-Forth Method in <math>\omega</math>-Saturated Models</b>	<b>55</b>
5.1	Spaces of Types . . . . .	55
5.2	$\omega$ -Saturated Models . . . . .	57
5.3	Quantifier Elimination . . . . .	60
5.4	Historic and Bibliographic Notes . . . . .	63
<b>6</b>	<b>Examples Illustrating the Back-and-Forth Method</b>	<b>64</b>
6.1	Algebraically Closed Fields . . . . .	64
6.2	Differentially Closed Fields . . . . .	70
6.3	Boolean Algebras . . . . .	78
6.4	Ultrametric Spaces . . . . .	86
6.5	Modules and Existentially Closed Modules . . . . .	91
6.6	Real Closed Fields (not in the original edition) . . . . .	98
6.7	Historic and Bibliographic Notes . . . . .	105
<b>7</b>	<b>Arithmetic</b>	<b>108</b>
7.1	The Successor Function . . . . .	108
7.2	The Order . . . . .	110
7.3	The Sum . . . . .	111
7.4	Sum and Product: Coding of Finite Sets . . . . .	116
7.5	Coding of Formulas; Tarski's Theorem . . . . .	122
7.6	The Hierarchy of Arithmetic Sets . . . . .	124
7.7	Some Axioms, Models, and Fragments of Arithmetic . . . . .	134
7.8	Nonstandard Models with Arithmetic Definitions . . . . .	141
7.9	Arithmetic Translation of Henkin's Method . . . . .	142
7.10	The Notion of Proof; Decidable Theories . . . . .	147
7.11	Gödel's Theorem . . . . .	151
7.12	A Little Mathematical Fiction . . . . .	155
7.13	Historic and Bibliographic Notes . . . . .	158
<b>8</b>	<b>Ordinals and Cardinals</b>	<b>160</b>
8.1	Well-Ordered Sets . . . . .	160
8.2	Axiom of Choice . . . . .	164
8.3	Cardinals . . . . .	171
8.4	Cofinality . . . . .	177
8.5	Historic and Bibliographic Notes . . . . .	180

<b>9 Saturated Models</b>	<b>181</b>
9.1 Svenonius's Theorem . . . . .	183
9.2 Compact, Saturated, Homogeneous, and Universal Models	186
9.3 Resplendent Models . . . . .	191
9.4 Properties Preserved Under Interpretation . . . . .	195
9.5 Recursively Saturated Models . . . . .	197
9.6 Historic and Bibliographic Notes . . . . .	202
<b>10 Prime Models</b>	<b>204</b>
10.1 Omitting Types Theorem . . . . .	204
10.2 Prime Models, Atomic Models: The Denumerable Case .	207
10.3 Theories with Finitely Many Denumerable Models . . . .	209
10.4 Constructed Models . . . . .	212
10.5 Minimal Models . . . . .	215
10.6 Nonuniqueness of the Prime Model . . . . .	218
10.7 Historic and Bibliographic Notes . . . . .	223
<b>11 Heirs</b>	<b>225</b>
11.1 Heirs . . . . .	225
11.2 Definable Types . . . . .	230
11.3 End Extension Types in Arithmetic . . . . .	231
11.4 Stable Types and Theories . . . . .	233
11.5 Historic and Bibliographic Notes . . . . .	236
<b>12 Special Sons, Morley Sequences</b>	<b>239</b>
12.1 Special Sons . . . . .	239
12.2 Coheirs . . . . .	243
12.3 Morley Sequences . . . . .	246
12.4 The Independence Property . . . . .	249
12.5 Indivisible Morley Sequences . . . . .	255
12.6 An Example: The Theories of Chains . . . . .	262
12.7 Special Sequences . . . . .	268
12.8 Instability and Order . . . . .	270
12.9 Appendix: Ramsey's Theorem . . . . .	273
12.10 Historic and Bibliographic Notes . . . . .	275
<b>13 The Fundamental Order</b>	<b>277</b>
13.1 The Fundamental Order . . . . .	277
13.2 Stability Spectrum . . . . .	281
13.3 Some Examples . . . . .	285
13.4 Historic and Bibliographic Notes . . . . .	289
<b>14 Stability and Saturated Models</b>	<b>290</b>
14.1 Existence Theorem . . . . .	290
14.2 Nonexistence Theorems . . . . .	291

14.3	Resplendent Models . . . . .	294
14.4	Sufficiently Saturated Extensions of a Given Model . . .	295
14.5	Historic and Bibliographic Notes . . . . .	298
<b>15</b>	<b>Forking</b>	<b>299</b>
15.1	The Theorem of the Bound . . . . .	300
15.2	Forking and Nonforking Sons . . . . .	303
15.3	Multiplicity . . . . .	305
15.4	Stable Types in an Unstable Theory . . . . .	307
15.5	Historic and Bibliographic Notes . . . . .	308
<b>16</b>	<b>Strong Types</b>	<b>309</b>
16.1	The Finite Equivalence Relation Theorem . . . . .	309
16.2	Spaces of Strong Types; Open Mapping Theorem . . . .	312
16.3	Morley Sequences for Strong Types; Saturated Models Revisited . . . . .	314
16.4	Imaginary Elements . . . . .	318
16.5	Elimination of Imaginaries . . . . .	321
16.6	A Galois Theory for Strong Types . . . . .	328
16.7	Historic and Bibliographic Notes . . . . .	331
<b>17</b>	<b>Notions of Rank</b>	<b>332</b>
17.1	Lascar Rank . . . . .	332
17.2	Shelah Rank . . . . .	336
17.3	Morley Rank . . . . .	341
17.4	Local Ranks . . . . .	345
17.5	Historic and Bibliographic Notes . . . . .	349
<b>18</b>	<b>Stability and Prime Models</b>	<b>351</b>
18.1	Uniqueness Theorem . . . . .	351
18.2	Prime Models of a Totally Transcendental Theory . . . .	353
18.3	Galois Theory of Differential Equations . . . . .	358
18.4	Prime $ T ^+$ -Saturated Models . . . . .	365
18.5	Ehrenfeucht Models . . . . .	367
18.6	Two-Cardinal Theorem; $\aleph_1$ -Categorical Theories . . . .	370
18.7	Historic and Bibliographic Notes . . . . .	372
<b>19</b>	<b>Stability, Indiscernible Sequences and Weights</b>	<b>374</b>
19.1	Indiscernible Sequences . . . . .	374
19.2	Lascar Inequalities . . . . .	376
19.3	Weight of a Superstable Type . . . . .	381
19.4	Independence and Domination . . . . .	384
19.5	Historic and Bibliographic Notes . . . . .	392

<b>20 Dimension in Models of a Totally Transcendental Theory</b>	<b>393</b>
20.1 Rudin–Keisler Order . . . . .	393
20.2 Dimensional Types and Theories . . . . .	402
20.3 Classification of the Models of a Dimensional Theory . . . . .	409
20.4 The Dope . . . . .	414
20.5 Depth and the Main Gap . . . . .	416
20.6 Historic and Bibliographic Notes . . . . .	417
<b>Bibliography</b>	<b>419</b>
<b>Index of Notation</b>	<b>429</b>
<b>Index</b>	<b>433</b>