

Contents

Preface	xi
Foreword	xiii
Chapter 1. Set Theory and Logic	1
1. Set Theory and Logic	1
Elementary propositional logic	2
Equivalence, negation, “and,” “or,” truth tables	2
de Morgan’s Rules	3
Implication	4
Quantors	5
“all,” “exists”	6
Basic set theoretical concepts	7
Complement, union, intersection	9
Cartesian product	11
Definition of a function	12
Injective, surjective, bijective functions	14
Inverse function	15
Hilbert’s Hotel	16
Cardinality of sets	16
Countable sets	17
The Cantor-Russel Theorem	18
An uncountable set	18
The Characteristic function of a set	19
The Bernstein Equivalence Theorem	20
2. Order	22
Totally ordered sets	23
Bounds, lower and upper bounds, maximum, minimum	25
Suprema and infima, least upper bound	26
The Characterisation Theorem for Sups	27
The Least Upper Bound Axiom	28
Intervals	29
Theorem on the Classification of Intervals	30
3. Arithmetics	31
Addition	31
Axioms for addition	32
Multiplication, axioms for multiplication	33
Distributivity, definition of a field	35
4. Order and Arithmetic	37
Definition of an ordered field	37

The absolute value, the sign	40
Properties of the norm	41
The definition of the Real Number Field	42
Natural, entire, rational and irrational numbers	43
5. Basic Properties of the Real Number Field	45
The Archimedean Axiom proved	45
The density of the rationals	46
Every set of natural numbers has a minimum	46
Induction Principle proved	48
The Bernoulli Inequality	49
6. The Field of Complex Numbers	51
The definition of the Complex Number field	52
Quadratic equations solved	54
Complex conjugation	55
Absolute value	55
7. Metric Spaces	57
Definition of a metric space	58
Bounded sets in metric spaces	59
Bounded functions in metric spaces	60
Postscript	60

Kapitel 1. Deutsche Zusammenfassung: Zahlen 65

Chapter 2. Sequences, Infinite Series, and Convergence 92

1. Sequences and their Convergence	92
Example: Growth	93
Infinite series	93
The partial sums of the Geometric Series	95
Achilles and the Tortoise	96
Growth revisited	97
The Binomial Formula	98
Monotone sequences	98
Convergence in a metric space, limits	100
Notation for the limit, convergence–divergence	102
Convergence of infinite series	104
The Harmonic Series, the Geometric Series	105
The Monotone Convergence Theorem	106
Euler’s Number e	107
Convergence of series with nonnegative summands	108
Leibniz Criterion	108
Alternating Harmonic Series	109
Comparison Criterion	109
Uncountability of the set of real numbers	111
Decimal Expansions	112
Adding and multiplying sequences	114

The Cauchy Criterion	117
Definition of Cauchy sequence in a metric space	117
Cauchy Criterion formulated	118
Definition of \limsup	118
Cauchy Criterion for infinite series	120
Absolute convergence of infinite series	120
The Root Test	122
The Ratio Test	123
2. Real and Complex Power Series	123
Definition of a power series	124
Radius of Convergence, convergence of a power series	125
The functions \exp , \sin , and \cos	127
The product of two infinite series revisited	128
Convergence of the convolution	129
The exponential function: I	130
The functional equation of the exponential function	130
The addition theorem for \sin and \cos	130
Euler's formula for the exponential function	132
Irrationality of e	134
Postscript	134

Kapitel 2. Deutsche Zusammenfassung:	
Folgen, unendliche Reihen, Konvergenz	138

Chapter 3. Functions of One Variable: Continuity	161
1. Metric Spaces	161
Accumulation points	161
Closed sets, open sets	163
Topology, topological space, topological property	165
Order topology, metric topology, discrete and indiscrete topologies ..	165
Connected metric spaces	167
Topological Characterisation of Real Intervals	167
2. Continuous functions	169
Definition of Continuity	169
Continuous functions and connectivity	172
The Intermediate Value Theorem	173
Handling continuous functions	174
Continuity of power series functions	178
The exponential function: II	178
The real logarithm	179
The exponential function: III	181
Angles	188
The argument function	188
The arc sine and arc cosine functions	189
The complex logarithm	190

Cluster points of sequences	191
Compact Metric spaces	193
Characterisation of Compact Real Sets	194
Maxima of Compact Real Sets	194
Continuous functions of compactness	196
Theorem of the Minimum and Maximum	196
Monotone functions and their continuity	197
Continuity of Monotone Functions	199
Convex functions and their continuity	201
Postscript	202

Kapitel 3. Deutsche Zusammenfassung: Funktionen einer Variablen: Stetigkeit 206

Chapter 4. Functions of One Variable: Differentiability 229

1. Definitions and Rules of Differentiation	229
Definition of differentiability	230
Various notations for the derivative	232
Sum rule and product rule	238
Differentiability of power series functions	240
Higher derivatives of power series functions	242
The chain rule	243
The quotient rule	245
Derivative of the inverse function	246
Derivative of the real logarithm	249
2. The Mean Value Theorem and its Consequences	251
Local extremal values	252
Rolle's Theorem	253
The Mean Value Theorem of Differential Calculus	254
Derivatives take intermediate values	255
Monotonicity of differentiable functions	256
Théorème d'accroissements finis	258
Path connectivity versus connectivity	259
Uniqueness of antiderivatives up to constants	261
The reciprocal has no antiderivative on $\mathbb{C} \setminus \{0\}$	261
Trigonometric functions revisited	263
The inverse functions of real trigonometric functions	264
The hyperbolic functions	265
Tables of antiderivatives	267
The exponential function: IV—complex logarithm	268
Abel's Limit Theorem	270
The Generalized Means Value Theorem	271
Bernoulli–de l'Hôpital	272
Convexity of functions	272
Points of inflection	276

Taylor's Theorem	276
Smooth functions	278
Smooth functions which are not analytic	279
exp, sin, and cos are analytic	279
Postscript	281

**Kapitel 4. Deutsche Zusammenfassung:
Funktionen einer Variablen: Differenzierbarkeit 282**

Chapter 5. Functions of One Variable: Integrability ... 313

1. Definition of Integrability, Basics	314
Step functions	314
Area below the graph of a step function	316
Integral of a step function	317
Inequality of Cauchy and Schwarz	320
Definition of the Riemann-integral	320
Riemann's Integrability Criterion	321
2. The Fundamental Theorem	324
The Fundamental Theorem of Differential and Integral Calculus	327
Integrating powerseries functions	328
The smooth camel hump	329
Zero sets	329
The Cantor Set	330
The Cantor-Carathéodory Function	332
Characterisation of Riemann Integrability	333
3. The Rules of Integration	336
The Substitution Theorem	336
The area of the unit disk	339
The exponential function: V	340
Inhomogeneous linear differential equations solved	342
Integration by parts	342
Antiderivatives of rational functions	343
4. Improper integration	347
Infinite series and improper integrals	348
The integral criterion	349
The zeta function	350
The Gamma function	351
The Gauss error function	351
5. Polar coordinates and computation of areas	352
Area bounded by graphs of functions in polar coordinates	355
Postscript	355

Kapitel 5. Deutsche Zusammenfassung: Funktionen einer Variablen: Integrierbarkeit	359
Index of Symbols	387
Alphabetical Index	389
The End	398