
Contents

Introduction	vii
Preface	xi
1 Matroids and Flag Matroids	1
1.1 Matroids	1
1.1.1 Definition in terms of bases	2
1.1.2 Examples	2
1.1.3 Circuits	4
1.2 Representable matroids	5
1.3 Maximality Property	7
1.4 Increasing Exchange Property	9
1.5 Sufficient systems of exchanges	10
1.5.1 Strong Exchange Property	11
1.6 Matroids as maps	12
1.7 Flag matroids	12
1.7.1 Flags	12
1.7.2 Flag matroids	13
1.7.3 Matroid quotients	13
1.7.4 Equivalence of Maximality Property and concordance of constituents	14
1.7.5 Representable flag matroids	15
1.7.6 Higgs lift	17
1.8 Flag matroids as maps	18
1.9 Exchange properties for flag matroids	19
1.9.1 Increasing Exchange Property for flag matroids	19
1.9.2 Failure of the Strong Exchange Property for flag matroids ..	19
1.10 Root system	20
1.10.1 Roots	20
1.10.2 Transpositions and reflections	21
1.10.3 Geometric representation of flags	22

1.10.4	Orderings associated with the root system	23
1.11	Polytopes associated with flag matroids	24
1.11.1	Polytopes associated with flag matroids	24
1.11.2	Main Theorem	25
1.12	Properties of matroid polytopes	27
1.12.1	Adjacency in matroids	27
1.12.2	Groups generated by transpositions	27
1.12.3	Components of matroids and the transposition graph	28
1.12.4	2-dimensional faces of matroid polytopes	29
1.12.5	Dimension of the matroid polytope	30
1.13	Minkowski sums	30
1.14	Exercises for Chapter 1	33
2	Matroids and Semimodular Lattices	37
2.1	Lattices as generalizations of projective geometry	38
2.2	Semimodular lattices	38
2.3	Jordan–Hölder permutation	39
2.4	Geometric lattices	42
2.4.1	Bases of lattices	42
2.4.2	Closure operators	43
2.4.3	Geometric lattice determined by a matroid	43
2.5	Representations of matroids	44
2.6	Representation of flag matroids	47
2.6.1	Retractions	48
2.6.2	Matroid maps from chains	49
2.7	Every flag matroid is representable	50
2.8	Exercises for Chapter 2	52
3	Symplectic Matroids	55
3.1	Definition of symplectic matroids	55
3.1.1	Hyperoctahedral group and admissible permutations	55
3.1.2	Admissible orderings	56
3.1.3	Symplectic matroids	57
3.2	Root systems of type C_n	58
3.2.1	Roots	58
3.2.2	Simple systems of roots	58
3.2.3	Correspondences	59
3.3	Polytopes associated with symplectic matroids	60
3.3.1	Geometric representation of admissible sets	60
3.3.2	Gelfand–Serganova Theorem for symplectic matroids	61
3.4	Representable symplectic matroids	63
3.4.1	Isotropic subspaces	63
3.4.2	Symplectic matroids from isotropic subspaces	64
3.4.3	Examples	65
3.4.4	Operations on representations	66

3.5	Homogeneous symplectic matroids	67
3.6	Symplectic flag matroids	69
3.6.1	Examples	70
3.6.2	Representable symplectic flag matroids	71
3.7	Greedy Algorithm	73
3.8	Independent sets	74
3.9	Symplectic matroid constructions	75
3.10	Orthogonal matroids	75
3.10.1	D_n -admissible orderings	75
3.10.2	Orthogonal matroids	76
3.10.3	Representable orthogonal matroids	77
3.10.4	Orthogonal flag matroids	77
3.11	Open problems	77
3.12	Exercises for Chapter 3	78
4	Lagrangian Matroids	81
4.1	Lagrangian matroids	81
4.1.1	Transversals	81
4.1.2	Symmetric Exchange Axiom	82
4.1.3	Represented Lagrangian matroids	83
4.1.4	Homogeneous Lagrangian matroids	84
4.2	Circuits and strong exchange	84
4.2.1	Dual matroid	84
4.2.2	Circuits	85
4.2.3	Circuits and cocircuits	86
4.2.4	Strong Exchange Property	87
4.2.5	Circuit characterizations of Lagrangian matroids	88
4.3	Maps on orientable surfaces	91
4.3.1	Maps on compact surfaces	91
4.3.2	Matroids, representations and maps	92
4.4	Exercises for Chapter 4	98
5	Reflection Groups and Coxeter Groups	101
5.1	Hyperplane arrangements	101
5.1.1	Chambers of a hyperplane arrangement	101
5.1.2	Galleries	103
5.2	Polyhedra and polytopes	105
5.3	Mirrors and reflections	106
5.3.1	Systems of mirrors and of reflections	107
5.3.2	Finite reflection groups	108
5.4	Root systems	109
5.4.1	Mirrors and their normal vectors	109
5.4.2	Root systems	110
5.4.3	Positive and simple systems	111
5.4.4	Classification of root systems	112

5.5	Isotropy groups	113
5.6	Parabolic subgroups	113
5.7	Coxeter complex	114
5.7.1	Chambers	114
5.7.2	Generation by simple reflections	116
5.7.3	Action of W on \mathcal{W}	117
5.8	Labeling of the Coxeter complex	117
5.9	Galleries	118
5.9.1	Bending	120
5.10	Generators and relations	122
5.10.1	Coxeter group	122
5.11	Convexity	123
5.12	Residues	125
5.12.1	The mirror system of a residue	126
5.12.2	Residues are convex	127
5.12.3	Gate property of residues	128
5.12.4	Opposite chamber in a residue	129
5.13	Foldings	129
5.14	Bruhat order	130
5.14.1	Characterization of the Bruhat order	131
5.14.2	Bruhat ordering on W/W_J	133
5.15	Splitting the Bruhat order	135
5.15.1	Some properties of the length function $l(w)$	135
5.15.2	The property Z	136
5.16	Generalized permutahedra	138
5.17	Symmetric group as a Coxeter group	141
5.17.1	Coxeter complex of the symmetric group	141
5.17.2	Permutahedron	142
5.17.3	Length in Sym_n	142
5.17.4	Bruhat order in Sym_n	143
5.18	Exercises for Chapter 5	144
6	Coxeter Matroids	151
6.1	Coxeter matroids	151
6.1.1	The Maximality Property	152
6.1.2	Matroid maps	152
6.1.3	Flag matroids are Coxeter matroids	153
6.1.4	The Strong Exchange Property	154
6.1.5	The Increasing Exchange Property	154
6.2	Root systems	155
6.2.1	Orbits of W on V	155
6.2.2	Orderings of $W \cdot \omega_J$	156
6.3	The Gelfand–Serganova Theorem	157
6.3.1	A Useful reformulation of the Gelfand–Serganova Theorem	159
6.3.2	A corollary	159

6.4	Coxeter matroids and polytopes	159
6.5	Examples	160
6.6	W -matroids	161
6.7	Characterization of matroid maps	168
6.8	Adjacency in matroid polytopes	169
6.9	Combinatorial adjacency	170
6.10	The matroid polytope	172
6.11	Exchange groups of Coxeter matroids	174
6.11.1	Dimension of the matroid polytope	175
6.12	Flag matroids and concordance	175
6.12.1	Shifts	176
6.12.2	Concordance	177
6.12.3	Constituents of a flag matroid	178
6.13	Combinatorial flag variety	179
6.13.1	Definition of the combinatorial flag variety	179
6.13.2	Weak map ordering	181
6.13.3	Expansion	181
6.14	Shellable simplicial complexes	183
6.15	Shellability of the combinatorial flag variety	186
6.16	Open problems	187
6.17	Exercises for Chapter 6	189
7	Buildings	199
7.1	Gaussian decomposition	199
7.2	BN -pairs	202
7.2.1	Definition of a BN -pair	202
7.2.2	Standard generators are involutions	203
7.2.3	Length function	203
7.2.4	Bruhat decomposition	204
7.2.5	Refinement of Axiom $BN1$	205
7.3	Deletion Property	206
7.4	Deletion property and Coxeter groups	208
7.5	Reflection representation of W	211
7.5.1	Construction	211
7.5.2	The Coxeter graph	213
7.5.3	Irreducibility of the reflection representation	213
7.5.4	Finite Coxeter groups are Euclidean reflection groups	214
7.5.5	Positive and negative roots	215
7.5.6	The reflection representation is faithful	215
7.6	Classification of finite Coxeter groups	216
7.6.1	Labeled graphs and associated bilinear forms	216
7.6.2	Classification of positive definite graphs	216
7.7	Chamber systems	220
7.7.1	Chamber systems	220
7.7.2	Coxeter complex	220

7.7.3	Residues and parabolic subgroups	220
7.7.4	The geometric realization	221
7.7.5	Flag complex of a vector space	222
7.8	W -metric	223
7.8.1	W -metrics and associated chamber systems	223
7.8.2	Order complex of a semimodular lattice admits a W -metric	224
7.9	Buildings	226
7.9.1	Definition of buildings	226
7.9.2	Generalized m -gons	226
7.9.3	Buildings of projective spaces	228
7.9.4	Building associated with a BN -pair	230
7.9.5	Strongly transitive automorphism groups	231
7.10	Representing Coxeter matroids in buildings	233
7.10.1	Retractions	233
7.10.2	Apartments are convex	234
7.10.3	Geodesic galleries and reduced words	235
7.10.4	Retractions give matroid maps	236
7.11	Vector-space representations and building representations	237
7.11.1	A_n , B_n , C_n and D_n -representations	237
7.11.2	Buildings from flags of subspaces	238
7.11.3	Vector-space representations of W -matroids are building representations	239
7.12	Residues in buildings	240
7.12.1	Residues are convex	240
7.12.2	Residues are buildings	240
7.12.3	Intersection of residues	241
7.12.4	Intersection of a residue and an apartment	241
7.13	Buildings of type $A_{n-1} = \text{Sym}_n$	241
7.14	Combinatorial flag varieties, revisited	243
7.14.1	Gaussian schemes	243
7.14.2	Retractions	245
7.14.3	Representation morphism	245
7.14.4	Partial metric on Ω_W^*	246
7.14.5	The case $W = A_{n-1}$	248
7.15	Open Problems	248
7.16	Exercises for Chapter 7	250
References		253
Index		259