

CONTENTS

Preface	x
Notation	xii
Chapter 1. Symplectic geometry in Euclidean space	
1.1 Some information from matrix group theory	1
1.1.1 Lie groups and algebras	1
1.1.2 The complete linear groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ and their Lie algebras	3
1.1.3 The special linear groups $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$	4
1.1.4 The orthogonal group $O(n)$ and the special orthogonal group $SO(n)$	6
1.1.5 The unitary group $U(n)$ and the special unitary group $SU(n)$	10
1.1.6 Connected components of matrix groups	13
1.1.7 The realification operation and complex structures	16
1.2 Groups of symplectic transformations of a linear space	24
1.2.1 Symplectic linear transformations	24
1.2.2 The noncompact groups $Sp(n, \mathbb{R})$ and $Sp(n; \mathbb{C})$	29
1.2.3 The compact group $Sp(n)$	39
1.2.4 The relation between symplectic groups and other matrix groups	46
1.3 Lagrangian manifolds	51
1.3.1 Real Lagrangian manifolds in a symplectic linear space	51
1.3.2 Complex Lagrangian Grassmann manifolds	59
1.3.3 Real Lagrangian Grassmann manifolds	68
Chapter 2. Symplectic geometry on smooth manifolds	
2.1 Local structure of symplectic manifolds	74
2.1.1 Local symplectic coordinates	74
2.1.2 Hamiltonian vector fields	77
2.1.3 The Poisson bracket	81
2.1.4 Darboux' theorem	87

2.2	Embeddings of symplectic manifolds	91
2.2.1	Embeddings of symplectic manifolds in \mathbb{R}^{2N}	91
2.2.2	Embeddings of symplectic manifolds in CP^N	93
2.2.3	Examples of symplectic manifolds	94

Chapter 3. Hamiltonian systems with symmetries on symplectic manifolds

3.1	Liouville's theorem	99
3.1.1	Integrals of Hamiltonian systems	99
3.1.2	Complete involutive sets of functions	101
3.2	Hamiltonian systems with noncommutative symmetries	107
3.2.1	Finite-dimensional Lie subalgebras in a space of functions on a symplectic manifold	107
3.2.2	A theorem on integration of systems with noncommutative symmetries	110
3.2.3	Connections between systems with commutative and noncommutative symmetries	114
3.2.4	Noncommutative integration in those cases when the sets of integrals do not form an algebra	120
3.2.5	Integration in quadratures of systems with noncommutative integrals	124
3.2.6	The canonical form of the Poisson bracket in a neighbourhood of a singular point. The case of degenerate Poisson brackets	131
3.2.7	Noncommutative integrability and its connection with canonical submanifolds and isotropic tori	138
3.2.8	Solvable Lie algebras of functions on symplectic manifolds and integration of mechanical systems corresponding to them	154
3.3	Dynamical systems generated by sectional operators	160
3.3.1	General plan of construction of sectional operators	160
3.3.2	Construction of a many-parameter family of exterior 2-forms on orbits of stationary groups of symmetric spaces	163

Chapter 4. Geodesic flows on two-dimensional Riemann surfaces

4.1	Completely integrable geodesic flows on a sphere and a torus	175
4.1.1	Geodesic flow of a two-dimensional Riemannian metric	175
4.1.2	A necessary and sufficient condition for the existence of an additional polynomial integral quadratic in the momenta	176
4.1.3	Description of Riemannian metrics on a sphere and a torus that admit an additional integral	182
4.1.4	Geometric properties of metrics on a sphere that admits an additional integral	187
4.2	Nonintegrability of analytic geodesic flows on surfaces of genus $g > 1$	191
4.3	Nonintegrability of the problem of n centres for $n > 2$	194
4.4	Morse-type theory of integrable Hamiltonian systems. Connections between integrability of systems, existence of stable periodic solutions and the one-dimensional homology group of surfaces of constant energy	199

Chapter 5. Effective methods of constructing completely integrable systems on Lie algebras.

Dynamics of multi-dimensional rigid body

5.1	Left-invariant Hamiltonian systems on Lie groups and the Euler equations on Lie algebras	226
5.1.1	Symplectic structure and left-invariant Hamiltonians	226
5.1.2	Quadratic Hamiltonians associated with the displacement of the argument on Lie algebras	235
5.1.3	Properties of the general Euler equations	240
5.2	A brief summary of classical results on the root decomposition of complex semisimple Lie algebras	245
5.3	Analogs of multidimensional rigid body motion for semisimple Lie algebras	251
5.3.1	The sectional decomposition of an algebra coincides with Cartan's decomposition	251

5.3.2	Various types of sectional operators. Complex metrics. Normal nilpotent metrics. Normal solvable metrics	256
5.3.3	Compact series of metrics	259
5.3.4	Normal series of metrics	266
5.4	Construction of integrals of the Euler equations corresponding to complex, compact and normal dynamics of multi-dimensional rigid body	271
5.4.1	Integrals of a complex left-invariant metrics	271
5.4.2	Integrals of a compact left-invariant metrics	274
5.4.3	Integrals of a normal left-invariant metrics	282
5.4.4	Involutoriness of integrals	282
5.5	Complete integrability of the Euler equations for "symmetrical" multi-dimensional rigid body	287
5.5.1	Complex integrable cases	287
5.5.2	Compact integrable cases	303
5.5.3	Normal integrable cases	305
5.5.4	Integrability of the Euler equations on singular orbits	334
5.6	Quadratic integrals of the Euler equations	341
5.7	Integrability of geodesic flows of left-invariant metrics of the form φ_{abD} on semisimple groups and geodesic flows on symmetric spaces	351
5.7.1	Geodesic flow on $T^*\mathfrak{G}$	351
5.7.2	\mathfrak{G} -invariant geodesic flows on $T^*(\mathfrak{G}/\mathfrak{H})$	355
5.7.3	Geodesic flows of general form on symmetric surfaces	365

Chapter 6. A brief review of the theory of topological classification of integrable nondegenerate Hamiltonian equations with two degrees of freedom

6.1	Formulation of the problem	368
6.1.1	Example: classical Hamiltonian equations of the motion of a rigid body	368
6.1.2	Integrability or nonintegrability as a manifestation of symmetry or randomness in system evolution	372
6.1.3	Examples of physical and mechanical systems integrable in the Liouville sense	373

6.1.4	Classification of all integrable nondegenerate Hamiltonian systems (integrable Hamiltonians) with two degrees of freedom	376
6.2	Smooth functions typical on smooth manifolds	377
6.2.1	Morse simple functions	377
6.2.2	Simple atoms and simple molecules	379
6.2.3	Complex Morse functions	386
6.2.4	Complex atoms and complex molecules	387
6.3	Bott's functions as "typical" integrals of integrable systems	391
6.3.1	Bott's functions	391
6.3.2	Integrals which are "typical" in the Hamiltonian physics	393
6.4	Rough and fine topological equivalence of integrable systems	395
6.5	Theorem of rough and fine classification of integrable Hamiltonian systems with two degrees of freedom. Applications in physics and mechanics	398
6.5.1	Formulation of the main theorem	398
6.5.2	Relation between invariants W , W^* and the topology of an integrable system. Substantial interpretation of atoms and molecules	416
6.6	Method of computing topological invariants for specific physical integrable Hamiltonians	426
6.7	A brief historical commentary	431
6.8	Class (H) of isoenergy three-dimensional integrable manifolds. "Five faces" of this class	434
6.8.1	Class (H) of the isoenergy 3-surfaces	434
6.8.2	Class (Q) of three-dimensional manifolds glued from two types of blocks	435
6.8.3	Class (W) of Waldhausen manifolds (graph-manifolds)	436
6.8.4	The class (S)	437
6.8.5	The class (T) of isointegrable manifolds corresponding to Hamiltonians with tame integrals	437
6.8.6	The class (R) of manifolds glued from round handles	438
6.8.7	Theorem on the coincidence of five classes	439

6.9	Application of the topological classification theory of integrable systems to geodesic flows on a 2-sphere and 2-torus	441
6.9.1	Hypothesis on geodesic flows	441
6.9.2	Integrable geodesic flows on a 2-sphere and a 2-torus	442
6.9.3	Complexity of integrable geodesic flows on a 2-sphere and a 2-torus	445
6.9.4	Hypothesis: linearly-quadratically integrable metrics “approximate” any nondegenerate integrable Riemannian metric on a 2-torus	447
6.10	Topological classification of classical cases of integrability in the dynamics of a heavy rigid body	449
	References	454
	Subject Index	464