CONTENTS

1 Introduction 1

Minkowski space 2
The Penrose transform 7
The Penrose transform on flag varieties 9

2 Lie Algebras and Flag Manifolds 10

2.1 Some structure theory 10
2.2 Borel and parabolic subalgebras 13
2.3 Generalized flag varieties 14
2.4 Fibrations of generalized flag varieties 18

3 Homogeneous Vector Bundles on G/P 21

3.1 A brief review of representation theory 21
3.2 Homogeneous bundles on G/P 28
3.3 A remark on inverse images 33

4 The Weyl Group, its Actions, and Hasse Diagrams 34

4.1 The Weyl Group 34
4.2 The affine Weyl action 39
4.3 The Hasse diagram of a parabolic subalgebra 39
4.4 Relative Hasse diagrams 43

5 The Bott–Borel–Weil Theorem 44

5.1 A simple proof 45
5.2 Some examples 48
5.3 Direct images 50

6 Realizations of G/P 53

6.1 The projective realization 55
6.2 The cell structure of G/P 56
6.3 Integral cohomology rings 58
6.4 Co-Adjoint realizations and moment maps 61

7 The Penrose Transform in Principle 67

7.1 Pulling-back cohomology 69

7.2 Pushing-down cohomology 71
7.3 A spectral sequence 72

8 The Bernstein–Gelfand–Gelfand Resolution 74

8.1 A prototype 74
8.2 Translating BGG resolutions 78
8.3 The general case on G/B 79
8.4 The story for G/B 80
8.5 An algorithm for computation 83
8.6 Non-standard morphisms 84
8.7 Relative BGG resolutions 85

9 The Penrose Transform in Practice 87

9.1 The homogeneous Penrose transform 89
9.2 The real thing 90
Positive helicity *ZRM* fields: $k + 2 < 0$ 91
Negative helicity fields: $k + 2 > 0$ 95
The scalar wave equation: $k = -2$ 99
9.3 The Penrose transform of forms on twistor space 103
9.4 Other bundles on twistor space 105
Deformations and nonlinear gravitons 105
The Einstein bundle 106
9.5 The Penrose transform for ambitwistor space 107
9.6 Higher dimensions—conformal case 110
Zero rest mass fields in even dimensions 111
Zero rest mass fields in odd dimensions 115
9.7 A Grassmannian generalization 117
9.8 An exceptional example 123
9.9 The Ward correspondence 126
Background coupling 138

10 Constructing Unitary Representations 142

10.1 The discrete series of SU(1,1) 142
10.2 Massless field representations 147
10.3 The twistor point of view 149
10.4 The twistor transform 154
10.5 Hermitian symmetric spaces 155
10.6 Towards discrete series 156

11 Module Structures on Cohomology 158

11.1 Verma modules and differential operators 159
Verma modules 159
Relation to differential operators 160
Duals of Verma modules and jet sheaves 160
Elementary properties of Verma modules 162
Verma modules for other parabolics 163
11.2 Invariant differential operators 164
Characterization 164
Conformally invariant operators 165
General results 165
11.3 The algebraic Penrose transform 166
Categories of g–modules and homogeneous sheaves 167
Algebraic equivalent of cohomology 167
The algebraic transform and Zuckerman functors 168
Dolbeault resolution and $\Gamma_{\mathfrak{g}}^{i}$ 170
11.4 K-types, local cohomolgy, and elementary states 171
Elementary states 172
g-module structure on H_{∞} 173
Local cohomology 174
Koszul complexes and local cohomology 177
Explicit hyperfunction representatives 180
K-types and Grothendieck duality 182
11.5 Homomorphisms of Verma modules 185

12 Conclusions and Outlook 191

Index 205