

Contents

Introduction	7
I Smooth Lie group actions on manifolds	13
1 Generalities	13
1.1 Notation	13
1.2 Orbits and fundamental vector fields	14
1.3 Examples	15
1.4 More definitions	15
1.5 Equivariant maps and orbit spaces	16
2 Equivariant tubular neighborhoods and orbit types decomposition	17
2.1 The slice theorem (equivariant tubular neighborhood)	17
2.2 Applications	19
3 Examples: 2 and 3-dimensional S^1 -manifolds	22
3.1 S^1 -actions on surfaces	22
3.2 3-manifolds; the principal case	25
3.3 Seifert manifolds	28
3.4 Associated principal actions and Euler class	34
II Symplectic geometry	37
1 Symplectic manifolds	37
1.1 Symplectic vector spaces	37
1.2 Symplectic manifolds, definition	38
1.3 The Darboux theorem	39
1.4 Examples: $U(n)$ -orbits in hermitian matrices	41
1.5 Calibrated almost complex structures	42
2 Hamiltonian vector fields and Poisson manifolds	43
2.1 Hamiltonian vector fields	43
2.2 The Poisson bracket on a symplectic manifold	45
3 Symplectic and hamiltonian actions	46
3.1 Symplectic actions	46
3.2 Hamiltonian actions	47
3.3 A machine producing examples: the coadjoint representation of a Lie group	49
3.4 Some properties of moment maps	52
3.5 Noether type theorems	54

3.6	Symplectic reduction, examples	57
III	Morse theory for hamiltonians	61
1	Critical points of almost periodic hamiltonians	61
1.1	Almost periodic hamiltonians	61
1.2	Critical points	62
2	Morse functions (in the sense of Bott)	63
2.1	Definitions	63
2.2	Frankel's theorem	64
2.3	Perestroïka	64
3	Connectivity of the fibers of the moment map	66
3.1	Connectivity of levels	66
3.2	Case of almost periodic hamiltonians	66
4	Application to convexity theorems	66
4.1	Proof of the Hausdorff-Ginsburg theorem	66
4.2	Convexity of the image of the moment map	67
4.3	Application: a theorem of Schur on hermitian matrices	71
4.4	Application: a theorem of Kushnirenko on monomial equations	72
IV	About manifolds of this dimension	75
1	Characterisation of those circle actions which are hamiltonian	76
1.1	Statement of the theorem	76
1.2	Proof	76
2	Symplectic reduction of the regular levels for a periodic hamiltonian	78
2.1	What happens near an extremum	78
2.2	What happens when going through a critical value	79
2.3	First applications	80
2.4	What happens when there are only two critical values	81
3	Blowing up fixed points; creation of index 2 critical points	84
3.1	Blowing up 0 in \mathbb{C}^2	84
3.2	Extension of an S^1 -action	86
3.3	Gradient manifolds and exceptional divisors	87
4	4-manifolds with periodic hamiltonians	88
4.1	Description	88
5	Plumbing	90
5.1	Plumbing of disc bundles	90
5.2	Equivariant plumbing along star-shaped graphs	91
5.3	Periodic hamiltonians and plumbing	96
5.4	Description of W up to the maximum: finishing graphs	96
A	Appendix: compact symplectic $SU(2)$ -manifolds of dimension 4	99
A.1	A list of examples	99
A.2	Classification	101
B	Appendix: 4-dimensional S^1 -manifolds with no invariant symplectic form (examples)	104
B.1	Equivariant plumbing on non-simply connected graphs, examples	104

B.2	Generalisations	111
V	Equivariant cohomology and the Duistermaat-Heckman theorems	113
1	Principal and universal bundles	114
1.1	Principal bundles	114
1.2	Universal bundles	114
2	The Borel construction and equivariant cohomology	119
2.1	The Borel construction	119
2.2	Equivariant cohomology	120
2.3	Generators for de Rham cohomology	121
2.4	Euler classes for fixed point free T -actions	123
3	Equivariant cohomology and hamiltonian actions	125
3.1	Relationships between hamiltonian actions and equivariant cohomology	125
3.2	Variation of the reduced symplectic forms	126
4	Duistermaat-Heckman with singularities	128
4.1	The “simple” situation	128
4.2	General case of a fixed submanifold of signature $(2p, 2q)$	129
4.3	Application: The Duistermaat-Heckman problem at critical values	131
5	Localisation at fixed points	133
5.1	The support of a H^*BT -module	133
5.2	Supports of $H_T^*(U)$, examples	134
5.3	The localisation theorem	136
6	The Duistermaat-Heckman formula	139
6.1	The Duistermaat-Heckman formula	139
6.2	Examples of applications	140
A	Appendix: some algebraic topology	142
A.1	The Thom class of an oriented vector bundle	142
A.2	The Euler class of an oriented bundle, equivariant Euler class	143
A.3	The Gysin exact sequence	144
A.4	The cohomology of projective space	144
A.5	The Gysin homomorphism (case of an embedding)	144
A.6	The Gysin homomorphism: integration in the fibers	145
B	Appendix: various notions of Euler classes	146
B.1	The case of S^1 -bundles	147
B.2	Complex line bundles	148
VI	Toric manifolds	151
1	The action of $T_{\mathbb{C}}^N$ and its subgroups on \mathbb{C}^N	152
1.1	Nontrivial stabilizers	153
1.2	Subtori	153
1.3	Real and imaginary parts	154
2	Fans and toric varieties	154
2.1	Fans	154
2.2	Closing a fan, open subsets in \mathbb{C}^N , toric varieties	156

3	Fans, symplectic reduction, convex polyhedra	162
3.1	The moment map for the K -action	162
3.2	What is left from the $T_{\mathbb{C}}^N$ -action	164
4	Properties of the toric manifolds X_{Σ}	165
4.1	Compactness of X_{Σ}	165
4.2	Topology of X_{Σ} and classes of invariant symplectic forms	166
4.3	Integral polyhedra and invariant line bundles	167
4.4	The cohomology of X_{Σ}	171
5	Complex toric surfaces	172
5.1	Graphs associated with dimension 2 fans	172
5.2	Interpretation of the m_i	174
5.3	4-manifolds with hamiltonian T^2 -actions	176

References**177**