

Table
 \mathcal{D} -modules cohérents et holonomes

Introduction to algebraic theory of linear systems of differential equations

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