Contents

Preface, vii

Chapter 1
Introduction

1 **Examples and Prospectus.** *nomenclature*, 1

2 **Metric Spaces.** *elementary topological notions, an existence theorem for best approximations, compact sets, the Bolzano-Weierstrass theorem*, 3

3 **Normed Linear Spaces.** *the space $C[a,b]$, compact sets, completeness*, 7

4 **Inner-product Spaces.** *orthonormal sets, least-squares approximation, the Gram-Schmidt process*, 12

5 **Convexity.** *Carathéodory's theorem, the theorem on linear inequalities, Helly's theorem*, 16

6 **Existence and Unicity of Best Approximations.** *example of nonexistence, uniform convexity, strict convexity, continuity of best-approximation operator*, 20

7 **Convex Functions.** *local minima are absolute minima, continuity*, 24

Chapter 2
The Tchebycheff solution of inconsistent linear equations

1 **Introduction.** 28

2 **Systems of Equations with One Unknown.** *graphical solution, three algebraic algorithms*, 30

3 **Characterization of the Solution.** 34

4 **The Special Case,** $m = n + 1$. *hyperplanes, monotone norms, the explicit solution,* 37

5 **Pólya's Algorithm.** 42

6 **The Ascent Algorithm.** 45

7 **The Descent Algorithm.** 51

8 **Convex Programming.** 54

Chapter 3

Tchebycheff approximation by polynomials and other linear families

1 **Introduction.** 57

2 **Interpolation.** *the Lagrange formula, Vandermonde's matrix, the error formula, Tchebycheff polynomials, Hermite interpolation,* 58

3 **The Weierstrass Theorem.** *Bernstein polynomials, monotone operators, Fejér's theorem,* 65

4 **General Linear Families.** *characterization theorem, Haar conditions, alternation theorem, Markoff systems, theorem of de La Vallée Poussin,* 72

5 **The Unicity Problem.** *strong unicity theorem, Haar's theorem, Freud's theorem,* 80

6 **Discretization Errors: General Theory.** *the density of a set, the modulus of continuity of a function, convergence theorems,* 84

7 **Discretization: Algebraic Polynomials. The Inequalities of Markoff and Bernstein.** *convergence theorems,* 89

8 **Algorithms.** *statements and proofs of the two Remes algorithms,* 95

Chapter 4

Least-squares approximation and related topics

1 **Introduction.** *inner products, various orthogonal systems,* 101

2 **Orthogonal Systems of Polynomials.** *the recurrence relation, extremal property, numerical integration, Gaussian integration, Stieltjes' theorem,* 106

3 **Convergence of Orthogonal Expansions.** *uniform and least-squares convergence, Christoffel-Darboux identity, Bessel inequality, convergence of Fourier series, Fejér's theorem,* 115

4 **Approximation by Series of Tchebycheff Polynomials.** *Bernstein's "lethargy" theorem, Weierstrass' nondifferentiable function, estimates of the quantity* $E_n(f)$, 126

5 **Discrete Least-squares Approximation.** *various discrete orthogonality properties, further bounds on* $E_n(f)$, *the Erdös-Turán theorem*, 133

6 **The Jackson Theorems.** *the Favard result, various examples, the Dini-Lipschitz theorem*, 139

Chapter 5
Rational approximation

1 **Introduction.** *conversion of rational functions to continued fractions*, 150

2 **Existence of Best Rational Approximations.** *extensions of the classical theorem*, 153

3 **The Characterization of Best Approximations.** *three theorems for generalized rational approximations, an alternation theorem, the special case of ordinary rational functions*, 158

4 **Unicity; Continuity of Best-approximation Operators.** *unicity for generalized rational approximations, strong unicity, continuity and discontinuity of the operator*, 164

5 **Algorithms.** *the linear inequality method, the weighted minimax method, the differential correction method*, 169

6 **Padé Approximation and its Generalizations.** *existence, computation, generalization to other bases*, 173

7 **Continued Fractions.** *elementary operations, Seidel's convergence theorem, Stieltjes' theorem*, 180

Chapter 6
Some additional topics

1 **The Stone Approximation Theorem.** 190

2 **The Müntz Theorem.** *Gram's lemma*, 193

3 **The Converses of the Jackson Theorems.** *the theorems of Bernstein and Zygmund*, 199

4 **Polygonal Approximation and Bases in** $C[a,b]$. *a Schauder base for* $C[a,b]$, *the theorem of Krein, Milman, and Rutman*, 205

5 **The Kharshiladze-Lozinski Theorems.** *Baire's theorem, the uniform-boundedness theorem, the extremal property of the Fourier projection, the Lebesgue constants, theorems of Nicolaev, Faber, and Berman,* 210

6 **Approximation in the Mean.** *Jackson's unicity theorem, a characterization theorem, Markoff's theorem,* 218

Notes, 224

References, 234

Index, 257