
Contents

Preface	V
Convex Functions	1
1 Convex Functions of One Variable	2
1.1 Preliminaries	2
1.2 Continuity, Support and Differentiability	4
1.3 Convexity Criteria	12
1.4 Jensen's and Other Inequalities	12
1.5 Bohr and Mollerup's Characterization of Γ	16
2 Convex Functions of Several Variables	20
2.1 Continuity, Support and First-Order Differentiability, and a Heuristic Principle	20
2.2 Alexandrov's Theorem on Second-Order Differentiability ..	27
2.3 A Convexity Criterion	32
2.4 A Stone–Weierstrass Type Theorem	34
2.5 A Sufficient Condition of Courant and Hilbert in the Calculus of Variations	35
Convex Bodies	39
3 Convex Sets, Convex Bodies and Convex Hulls	40
3.1 Basic Concepts and Simple Properties	41
3.2 An Excursion into Combinatorial Geometry: The Theorems of Carathéodory, Helly and Radon	46
3.3 Hartogs' Theorem on Power Series	50
4 Support and Separation	52
4.1 Support Hyperplanes and Support Functions	52
4.2 Separation and Oracles	58
4.3 Lyapunov's Convexity Theorem	60
4.4 Pontryagin's Minimum Principle	64
5 The Boundary of a Convex Body	68
5.1 Smooth and Singular Boundary Points, Differentiability	68

	5.2	Extreme Points	74
	5.3	Birkhoff's Theorem on Doubly Stochastic Matrices	76
6		Mixed Volumes and Quermassintegrals	79
	6.1	Minkowski Addition, Direct Sums, Hausdorff Metric, and Blaschke's Selection Theorem	80
	6.2	Minkowski's Theorem on Mixed Volumes and Steiner's Formula	88
	6.3	Properties of Mixed Volumes	93
	6.4	Quermassintegrals and Intrinsic Volumes	102
7		Valuations	110
	7.1	Extension of Valuations	111
	7.2	Elementary Volume and Jordan Measure	118
	7.3	Characterization of Volume and Hadwiger's Functional Theorem	126
	7.4	The Principal Kinematic Formula of Integral Geometry	134
	7.5	Hadwiger's Containment Problem	140
8		The Brunn–Minkowski Inequality	141
	8.1	The Classical Brunn–Minkowski Inequality	142
	8.2	The Brunn–Minkowski Inequality for Non-Convex Sets	146
	8.3	The Classical Isoperimetric and the Isodiametric Inequality and Generalized Surface Area	147
	8.4	Sand Piles, Capillary Surfaces and Wulff's Theorem in Crystallography	155
	8.5	The Prékopa–Leindler Inequality and the Multiplicative Brunn–Minkowski Inequality	161
	8.6	General Isoperimetric Inequalities and Concentration of Measure	164
9		Symmetrization	168
	9.1	Steiner Symmetrization	168
	9.2	The Isodiametric, Isoperimetric, Brunn–Minkowski, Blaschke–Santaló and Mahler Inequalities	175
	9.3	Schwarz Symmetrization and Rearrangement of Functions .	178
	9.4	Torsional Rigidity and Minimum Principal Frequency	179
	9.5	Central Symmetrization and the Rogers–Shephard Inequality	185
10		Problems of Minkowski and Weyl and Some Dynamics	187
	10.1	Area Measure and Minkowski's Problem	188
	10.2	Intrinsic Metric, Weyl's Problem and Rigidity of Convex Surfaces	197
	10.3	Evolution of Convex Surfaces and Convex Billiards	199
11		Approximation of Convex Bodies and Its Applications	202
	11.1	John's Ellipsoid Theorem and Ball's Reverse Isoperimetric Inequality	203
	11.2	Asymptotic Best Approximation, the Isoperimetric Problem for Polytopes, and a Heuristic Principle	209

12	Special Convex Bodies	218
12.1	Simplexes and Choquet's Theorem on Vector Lattices	218
12.2	A Characterization of Balls by Their Gravitational Fields	222
12.3	Blaschke's Characterization of Ellipsoids and Its Applications	225
13	The Space of Convex Bodies	230
13.1	Baire Categories	231
13.2	Measures on \mathcal{C} ?	234
13.3	On the Metric Structure of \mathcal{C}	236
13.4	On the Algebraic Structure of \mathcal{C}	237
Convex Polytopes		243
14	Preliminaries and the Face Lattice	244
14.1	Basic Concepts and Simple Properties of Convex Polytopes	244
14.2	Extension to Convex Polyhedra and Birkhoff's Theorem	247
14.3	The Face Lattice	252
14.4	Convex Polytopes and Simplicial Complexes	257
15	Combinatorial Theory of Convex Polytopes	258
15.1	Euler's Polytope Formula and Its Converse by Steinitz for $d = 3$	259
15.2	Shelling and Euler's Formula for General d	265
15.3	Steinitz' Polytope Representation Theorem for $d = 3$	270
15.4	Graphs, Complexes, and Convex Polytopes for General d	272
15.5	Combinatorial Types of Convex Polytopes	277
16	Volume of Polytopes and Hilbert's Third Problem	280
16.1	Elementary Volume of Convex Polytopes	280
16.2	Hilbert's Third Problem	288
17	Rigidity	292
17.1	Cauchy's Rigidity Theorem for Convex Polytopal Surfaces	292
17.2	Rigidity of Frameworks	297
18	Theorems of Alexandrov, Minkowski and Lindelöf	301
18.1	Alexandrov's Uniqueness Theorem for Convex Polytopes	301
18.2	Minkowski's Existence Theorem and Symmetry Condition	303
18.3	The Isoperimetric Problem for Convex Polytopes and Lindelöf's Theorem	308
19	Lattice Polytopes	310
19.1	Ehrhart's Results on Lattice Point Enumerators	310
19.2	Theorems of Pick, Reeve and Macdonald on Volume and Lattice Point Enumerators	316
19.3	The McMullen–Bernstein Theorem on Sums of Lattice Polytopes	320
19.4	The Betke–Kneser Theorem on Valuations	324
19.5	Newton Polytopes: Irreducibility of Polynomials and the Minding–Kouchnirenko–Bernstein Theorem	332

20	Linear Optimization	335
20.1	Preliminaries and Duality	336
20.2	The Simplex Algorithm	339
20.3	The Ellipsoid Algorithm	343
20.4	Lattice Polyhedra and Totally Dual Integral Systems	345
20.5	Hilbert Bases and Totally Dual Integral Systems	348
Geometry of Numbers and Aspects of Discrete Geometry		353
21	Lattices	355
21.1	Basic Concepts and Properties and a Linear Diophantine Equation	356
21.2	Characterization of Lattices	359
21.3	Sub-Lattices	361
21.4	Polar Lattices	365
22	Minkowski's First Fundamental Theorem	366
22.1	The First Fundamental Theorem	366
22.2	Diophantine Approximation and Discriminants of Polynomials	370
23	Successive Minima	375
23.1	Successive Minima and Minkowski's Second Fundamental Theorem	376
23.2	Jarník's Transference Theorem and a Theorem of Perron and Khintchine	380
24	The Minkowski–Hlawka Theorem	385
24.1	The Minkowski–Hlawka Theorem	385
24.2	Siegel's Mean Value Theorem and the Variance Theorem of Rogers–Schmidt	388
25	Mahler's Selection Theorem	391
25.1	Topology on the Space of Lattices	391
25.2	Mahler's Selection Theorem	392
26	The Torus Group \mathbb{E}^d/L	395
26.1	Definitions and Simple Properties of \mathbb{E}^d/L	395
26.2	The Sum Theorem of Macbeath–Kneser	398
26.3	Kneser's Transference Theorem	403
27	Special Problems in the Geometry of Numbers	404
27.1	The Product of Inhomogeneous Linear Forms and DOTU Matrices	405
27.2	Mordell's Inverse Problem and the Epstein Zeta-Function	408
27.3	Lattice Points in Large Convex Bodies	410
28	Basis Reduction and Polynomial Algorithms	411
28.1	LLL-Basis Reduction	411
28.2	Diophantine Approximation, the Shortest and the Nearest Lattice Vector Problem	417

29 Packing of Balls and Positive Quadratic Forms 421

29.1 Densest Lattice Packing of Balls in Dimensions 2 and 3 422

29.2 Density Bounds of Blichfeldt and Minkowski–Hlawka 424

29.3 Error Correcting Codes and Ball Packing 427

29.4 Geometry of Positive Definite Quadratic Forms and Ball
Packing 430

30 Packing of Convex Bodies 439

30.1 Definitions and the Existence of Densest Lattice Packings . . 440

30.2 Neighbours 445

30.3 The Betke–Henk Algorithm and the Lower Bound
for $\delta_L(C)$ of Minkowski–Hlawka 448

30.4 Lattice Packing Versus Packing of Translates 450

31 Covering with Convex Bodies 454

31.1 Definitions, Existence of Thinnest Lattice Coverings
and the Covering Criterion of Wills 455

31.2 Star Numbers 457

31.3 Rogers’s Upper Bound for $\vartheta_T(C)$ 458

31.4 Lattice Covering Versus Covering with Translates 462

32 Tiling with Convex Polytopes 463

32.1 Dirichlet–Voronoi and Delone Tilings and Polyhedral
Complexes 463

32.2 The Venkov–Alexandrov–McMullen Characterization
of Translative Tiles 470

32.3 Conjectures and Problems of Voronoi, Hilbert, Minkowski,
and Keller 478

33 Optimum Quantization 480

33.1 Fejes Tóth’s Inequality for Sums of Moments 481

33.2 Zador’s Asymptotic Formula for Minimum Distortion 484

33.3 Structure of Minimizing Configurations, and a Heuristic
Principle 492

33.4 Packing and Covering of Circles, Data Transmission
and Numerical Integration 494

34 Koebe’s Representation Theorem for Planar Graphs 499

34.1 The Extension of Koebe’s Theorem by Brightwell
and Scheinerman 500

34.2 Thurston’s Algorithm and the Riemann Mapping Theorem . . 509

References 513

Index 567

Author Index 577