# Contents

**Preface** ..... v

**I. Preliminary Questions** ..... 1

1. Elements of topology ..... 1
2. Vectors and matrices ..... 9
3. Analytic functions of several variables ..... 21
4. Differentiable manifolds ..... 27

**II. Existence Theorems. General Properties of the Solutions** 29

1. Generalities ..... 29
2. The fundamental existence theorem ..... 30
*3. Continuity properties ..... 36
*4. Differentiability properties ..... 40
*5. Analyticity properties ..... 43
6. Equations of higher order ..... 45
*7. Autonomous systems ..... 46

**III. Linear Systems** ..... 55

1. Various types of linear systems ..... 55
2. Homogeneous systems ..... 57
3. Non-homogeneous systems ..... 68
4. Linear systems with constant coefficients ..... 69
5. Linear systems with periodic coefficients: Theory of Floquet ..... 73

**IV. Stability** ..... 76

1. Historical considerations ..... 76
2. Stability of critical points ..... 78
3. Stability in linear homogeneous systems ..... 79
4. Uniformly regular transformations ..... 80
5. Stability of trajectories ..... 83
6. Stability of mappings ..... 84
7. Further definitions of stability ..... 84

**V. The Differential Equation** ..... 86

$$\frac{dx}{dt} = Px + q(x;t)$$

($P$ a constant matrix; $q(0;t) = 0$)

1. General remarks ..... 87
2. The general non-analytic system ..... 88
*3. Analytic systems: Generalities ..... 95
*4. The expansion theorem of Liapunov ..... 96

**VI. The Differential Equation** ..... 107

$$\frac{dx}{dt} = Px + q(x;y)$$

($P$ a constant matrix; $q(0;t) = 0$)

(continued)

1. The method of Poincaré ..... 107
2. The direct stability theorems of Liapunov ..... 112
*3. Stability in product spaces ..... 122
*4. An existence theorem ..... 126
*5. Stability in product spaces: Analytical case ..... 130
*6. System with a single characteristic root zero and the rest with negative real parts ..... 133
7. The converse of Liapunov's theorems ..... 137

**VII. The Differential Equation** ..... 142

$$\frac{dx}{dt} = P(t)z + q(x;t)$$

($P(t)$ a variable matrix; $q(0;t) = 0$)

*1. Perron's reduction theorem ..... 142
*2. Various stability criteria ..... 145
*3. The Liapunov numbers. Application to stability ..... 152

**VIII. Periodic Systems and Their Stability** ..... 155

*1. Linear homogeneous systems with periodic coefficients ..... 155
*2. Analytic systems with periodic coefficients ..... 159
*3. Stability of periodic solutions ..... 160
*4. Stability of the closed paths of autonomous systems. The method of sections of Poincaré ..... 160
*5. Systems of periodic solutions ..... 164

*6. Quasi-linear systems and their periodic solutions.... 170
*7. A class of periodic solutions studied by Liapunov.... 172
*8. Complete families of periodic solutions..... 174

**IX. Two Dimensional Systems. Simple Critical Points. The Index. Behavior at Infinity**..... 181
1. Generalities..... 182
2. Critical points of linear homogeneous systems..... 183
3. Elementary critical points in the general case..... 188
4. The index. Application to differential equations..... 195
5. Behavior of the paths at infinity..... 201

**X. Two Dimensional Systems (continued)**..... 209
1. General critical points..... 209
2. Local phase-portrait at a critical point..... 214
3. The limiting sets of the paths as $t \to \pm \infty$..... 225
4. The theorem of Bendixson..... 230
5. Some complements on limit-cycles..... 235
6. On path-polygons..... 237
7. Some properties of div $(X, Y)$..... 238
8. Critical points with a single non-zero characteristic root..... 241
9. Structural stability..... 250
10. Non-analytical systems..... 257

**XI. Differential Equations of the Second Order**..... 264
1. Non-dissipative systems..... 266
2. Liénard's equation..... 267
3. The equation of van der Pol: Phase-portrait..... 272
4. The equation of Cartwright-Littlewood..... 279
5. Applications and complements..... 286
6. The differential equation $x'' + f(x, x')x' + g(x) = e(t)$..... 292
7. A special differential equation $x'' + g(x) = \mu \sin \omega t$.. 301
8. A special differential equation $x'' + f(x)x' + g(x) = e(t)$..... 306
9. Certain periodic systems investigated by Gomory..... 307

**XII. Oscillations in Systems of the Second Order. Methods of Approximation**..... 312
1. Self-excited systems..... 313
2. Forced oscillations..... 321
3. Approximations for quasi-harmonic systems..... 331
4. Equations of Mathieu and of Hill..... 337
5. The limiting position of limit-cycles..... 342

**Appendix I. Complement on Matrices** ..... 347
1. Reduction to normal form ..... 347
2. Normal form for real matrices ..... 352
3. Normal form of the inverse of a matrix ..... 354
4. Determination of log A ..... 355
5. A certain matrix equation ..... 356
6. Another matrix problem ..... 358

**Appendix II. Some Topological Complements** ..... 360
1. The index in the plane ..... 360
2. The index of a surface ..... 366
3. A property of planar Jordan curves ..... 370

**Problems** ..... 373

**Bibliography** ..... 376

**List of Principal Symbols** ..... 386

**Index** ..... 387