

# Contents

<b>Foreword</b>	v
<b>Preface</b>	vii
<b>I Integration Relative to Daniell Measures</b>	1
<b>1 Riesz Spaces</b>	3
1.1 Ordered Groups . . . . .	4
1.2 Riesz Spaces . . . . .	9
1.3 Order Dual of a Riesz Space . . . . .	12
1.4 Daniell Measures . . . . .	16
<b>2 Measures on Semirings</b>	26
2.1 Semirings, Rings, and $\sigma$ -Rings . . . . .	26
2.2 Measures on Semirings . . . . .	30
2.3 Lebesgue Measure on an Interval . . . . .	38
<b>3 Integrable and Measurable Functions</b>	40
3.1 Upper Integral of a Positive Function . . . . .	41
3.2 Convergence Theorems . . . . .	49
3.3 Integrable Sets . . . . .	56
3.4 $\sigma$ -Measurable Spaces . . . . .	60
3.5 Measurable Mappings . . . . .	62
3.6 Essentially Integrable Mappings . . . . .	66

3.7	Upper and Lower Integrals . . . . .	70
3.8	Atoms . . . . .	78
3.9	Prolongations of $\mu$ . . . . .	81
<b>4</b>	<b>Lebesgue Measure on <math>\mathbf{R}</math></b>	<b>86</b>
4.1	Base- $b$ Expansions of a Real Number . . . . .	86
4.2	The Cantor Singular Function . . . . .	88
4.3	Example of a Nonmeasurable Set . . . . .	90
<b>5</b>	<b><math>L^p</math> Spaces</b>	<b>93</b>
5.1	Definition of $L^p$ Spaces . . . . .	94
5.2	Convergence Theorems . . . . .	99
5.3	Convergence in Measure . . . . .	104
5.4	Uniformly Integrable Sets . . . . .	108
<b>6</b>	<b>Integrable Functions for Measures on Semirings</b>	<b>121</b>
6.1	Measurability . . . . .	121
6.2	Complements on the $L^p$ Spaces . . . . .	126
6.3	Measures Defined by Masses . . . . .	127
6.4	Prolongations of a Measure . . . . .	128
<b>7</b>	<b>Radon Measures</b>	<b>133</b>
7.1	Locally Compact Spaces . . . . .	134
7.2	Radon Measures . . . . .	137
7.3	Regularity of Radon Measures . . . . .	140
7.4	Lusin Measurable Mappings . . . . .	143
7.5	Atomic Radon Measures . . . . .	145
7.6	The Riemann Integral . . . . .	147
7.7	Weak Convergence . . . . .	153
7.8	Tight Sequences . . . . .	157
<b>8</b>	<b>Regularity</b>	<b>166</b>
8.1	Regular Measures . . . . .	166
<b>II</b>	<b>Operations on Measures Defined on Semirings</b>	<b>173</b>
<b>9</b>	<b>Induced Measures and Product Measures</b>	<b>175</b>
9.1	Measure Induced on a Measurable Set . . . . .	175
9.2	Fubini's Theorem . . . . .	178
9.3	Lebesgue Measure on $\mathbf{R}^k$ . . . . .	187
<b>10</b>	<b>Radon-Nikodym Derivatives</b>	<b>193</b>
10.1	Sums of Measures . . . . .	194
10.2	Locally Integrable Functions . . . . .	195
10.3	The Radon-Nikodym Theorem . . . . .	200

10.4	Combination of Operations on Measures . . . . .	205
10.5	Duality of $L^p$ Spaces . . . . .	207
10.6	The Yosida–Hewitt Decomposition Theorem . . . . .	210
<b>11</b>	<b>Images of Measures</b>	<b>224</b>
11.1	$\mu$ -Suited Pairs . . . . .	225
11.2	Infinite Product of Measures . . . . .	228
11.3	Change of Variable . . . . .	233
11.4	Elements of Ergodic Theory . . . . .	236
<b>12</b>	<b>Change of Variables</b>	<b>242</b>
12.1	Differentiation in $\mathbf{R}^k$ . . . . .	242
12.2	The Modulus of an Automorphism . . . . .	248
12.3	Change of Variables . . . . .	250
12.4	Polar Coordinates . . . . .	252
<b>13</b>	<b>Stieltjes Integral</b>	<b>265</b>
13.1	Functions of Bounded Variation . . . . .	266
13.2	Stieltjes Measures . . . . .	268
13.3	Line Integrals . . . . .	274
13.4	The Lebesgue Decomposition of a Function . . . . .	277
13.5	Upper and Lower Derivatives . . . . .	282
<b>14</b>	<b>The Fourier Transform in <math>\mathbf{R}^k</math></b>	<b>286</b>
14.1	Measures in $\mathbf{R}^k$ . . . . .	287
14.2	Distribution Functions . . . . .	293
14.3	Covariance Matrix . . . . .	294
14.4	The Fourier Transform . . . . .	296
14.5	Normal Laws in $\mathbf{R}^n$ . . . . .	303
<b>III</b>	<b>Convergence of Random Variables; Conditional Expectation</b>	<b>309</b>
<b>15</b>	<b>The Strong Law of Large Numbers</b>	<b>311</b>
15.1	Convergence in Probability . . . . .	312
15.2	Independence of Random Variables . . . . .	313
15.3	An Example of Independent Random Variables . . . . .	316
15.4	The One-Sided Shift Transformation . . . . .	318
15.5	Borel's Normal Number Theorem . . . . .	320
<b>16</b>	<b>The Central Limit Theorem</b>	<b>326</b>
16.1	Convergence in Law . . . . .	326
16.2	The Lindeberg Theorem . . . . .	330
16.3	The Central Limit Theorem . . . . .	334

<b>17</b>	<b>Order Statistics</b>	<b>344</b>
17.1	Definition of the Order Statistics . . . . .	344
17.2	Convergence of the Empirical Median . . . . .	347
<b>18</b>	<b>Conditional Probability</b>	<b>352</b>
18.1	Conditional Expectation . . . . .	353
18.2	The Converse of the Mean-Value Theorem . . . . .	358
18.3	Jensen's Inequality . . . . .	360
18.4	Conditional Expected Value Given a Random Variable . . . . .	363
18.5	Conditional Law of $Y$ Given $X$ . . . . .	364
18.6	Computation of Conditional Laws . . . . .	369
18.7	Existence of Conditional Laws when $G = \mathbf{R}^k$ . . . . .	371
<b>IV</b>	<b>Operations on Radon Measures</b>	<b>377</b>
<b>19</b>	<b><math>\mu</math>-Adequate Family of Measures</b>	<b>379</b>
19.1	Induced Radon Measure . . . . .	380
19.2	$\mu$ -Dense Families of Compact Sets . . . . .	382
19.3	Sums of Radon Measures . . . . .	385
19.4	$\mu$ -Adequate Families . . . . .	387
19.5	$\mu$ -Adapted Pairs . . . . .	391
<b>20</b>	<b>Radon Measures Defined by Densities</b>	<b>397</b>
20.1	Integration with Respect to Induced Measures . . . . .	397
20.2	Radon Measures with Base $\mu$ . . . . .	399
20.3	The Radon–Nikodym Theorem . . . . .	402
20.4	Duality of $L^p$ Spaces . . . . .	405
<b>21</b>	<b>Images of Radon Measures and Product Measures</b>	<b>408</b>
21.1	Images of Radon Measures . . . . .	408
21.2	Decomposition of a Measure in Slices . . . . .	410
21.3	Product of Radon Measures . . . . .	411
<b>22</b>	<b>Operations on Regular Measures</b>	<b>416</b>
22.1	Some Operations on Regular Measures . . . . .	416
22.2	Baire Sets . . . . .	419
22.3	Product of Regular Measures . . . . .	421
22.4	Change of Variable Formula . . . . .	423
<b>23</b>	<b>Haar Measures</b>	<b>425</b>
23.1	Invariant Measures . . . . .	426
23.2	Existence and Uniqueness of Left Haar Measure . . . . .	429
23.3	Modular Function on $G$ . . . . .	433
23.4	Relatively Invariant Measures on a Group . . . . .	434
23.5	Homogeneous Spaces . . . . .	436

23.6	Integration with Respect to $\lambda^\sharp$	442
23.7	Reconstitution of $\lambda_{/\beta}^\sharp$	444
23.8	Quasi-Invariant Measures on Homogeneous Spaces	446
23.9	Relatively Invariant Measures on $G/H$	449
23.10	Haar Measure on $SO(n+1, \mathbf{R})$	450
23.11	Haar Measure on $SH(n, \mathbf{R})$	454
<b>24</b>	<b>Convolution of Measures</b>	<b>465</b>
24.1	Convolvable Measures	466
24.2	Convolution of a Measure and a Function	467
24.3	Convolution of a Measure and a Continuous Function	471
24.4	Convolution of $\mu \in \mathcal{M}(G, \mathbf{C})$ and $f \in \overline{\mathcal{L}^p(\beta)}$	472
24.5	Convolution and Transposition	476
24.6	Convolution of Functions on a Group	479
24.7	Regularization	483
24.8	Definition of Gelfand Pair	488
<b>Index</b>		<b>499</b>
<b>Symbol Index</b>		<b>505</b>